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MMT-006

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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

June, 2011

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Question number **1** is **compulsory**. Do **any four** questions out of questions **2** to **7**. Calculators are **not** allowed.*

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1. Are the following statements *true* or *false* ? Justify your answer with the help of a short proof or a counter example. **5x2=10**

   (a) If E_1 and E_2 are open sets in a normed space X, then $E_1 + E_2$ is open in X.

   (b) Every continuous linear map is compact.

   (c) Every infinite dimensional space admits a discontinuous linear functional.

   (d) For any proper subspace Y of a normed space X, the interior Y° is empty.

   (e) Every normal operator on a normed space is unitary.

2. (a) State closed graph theorem and use the theorem to prove open mapping theorem. 5

(b) Let X = C [0, 1] with Sup norm defined by $\|f\| = \text{Sup}\{|f(x)|\}$. 3

$x \in [0, 1]$

Let T be a linear map defined on X by

$$T(f) = f\left(\frac{1}{2}\right).$$

Show that T is a bounded linear map such that $\|T\| = 1$.

(c) Let X = C′[0, 1] and Y = C [0, 11] and let $T : X \to Y$ be the linear operator from X to Y given by $T(f) = f'$, the derivative of f on [0, 1]. Show that T is not continuous. 2

3. (a) Let X be a finite dimensional normed space. Let E be a closed and bounded subset of X. Show that E is compact. 4

(b) Define Eigen Spectrum of a bounded linear operator on a Banach space. Show that the eigen spectrum of the operator T on l^2 given by $T(\propto_1, \propto_2 \ldots) = (0, \propto_1, \propto_2 \ldots..)$ is empty. 3

(c) Let $\|\cdot\|$ be a norm on a linear space X. If $x, y \in X$ and $\|x+y\| = \|x\| + \|y\|$, then show that $\|sx + ty\| = s\|x\| + t\|y\|$ for all $s \geq 0$, $t \geq 0$. 3

4. (a) State uniform boundedness principle. Use it and to show the following result. 4

Let X be a Banach space, Y be a normed space and $\{F_n\}$ be a sequence in B<(X, Y) such that the sequence $\{F_n(x)\}$ converges in Y for every $x \epsilon X$. For $x \epsilon X$, define.

$$F(x) = \lim_{n \to \infty} F_n(x)$$

Show that F is a bounded linear map from X to Y.

(b) Let X be an inner product space with the inner product given by <, >. For $x \epsilon X$, define the function $\|.\| : X \to K$ given by $\|x\| = <x, -x>^{1/2}$, the non negative square root of $<x, x>$. Show that $\|.\| : X \to K$ defines a norm on X and $|<(x, y)>| \leq \|x\| \, \|y\|$ for all $x, y \epsilon X$. Also show that for all $x, y \epsilon X$, 6

$$\|x+y\|^2 + \|x-y\|^2 = 2 \; (\|x\|^2 + \|y\|^2).$$

5. (a) Let X_1 be a closed subspace and X_2 be a finite dimensional subspace of a normed space X. Then show that $X_1 + X_2$ is closed in X. 3

(b) Let $X = L^2 \; [0, \; 2\pi]$ and $u_n(t) = \frac{e^{int}}{\sqrt{2\pi}}$, 3

$t \epsilon \{-\pi, \pi\}$, $n \epsilon Z$

Show that the set $E = \{u_1, u_2 \ldots\}$ is an orthonormal set in X.

(c) State Riesz Representation theorem. Let $H = \mathbb{C}^2$ and let $\int : H \to \mathbb{C}$ be defined by $= f(x_1, x_2) = x_2 - ix_1$. Find a $y \epsilon H$ that represents f. **4**

6. (a) Show that normed space is separable if its dual is separable. Is the converse true ? Justify your answer. **5**

(b) Let H be a Hilbert space. For any subset A of H, define $A^{\perp}$. If $A \subseteq B \subseteq H$, then show that : **5**

(i) $B^{\perp} \subseteq A^{\perp}$

(ii) $A \subseteq A^{\perp\perp}$

State conditions on A so that $A^{\perp\perp} = A$.

7. (a) Define a normal operator on a Hilbert space H. Show that an operator T on H is normal if and only if $\|Tx\| = \|T \forall x\| \ \forall x \epsilon H$. **5**

(b) Let X be a vector space. Let $\|.\|^1$ and $\|.\|^2$ be two norms on X. When are these norms said to be equivalent ? Justify your answer. **5**

Let $X = \mathbb{R}^3$. For $x = (x_1, x_2, x_3)$,

Let $\|x\|^1 = |x_1| + |x_2| + |x_3|$

$$\|x\|^2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$$

Show that $\|.\|^1$ and $\|.\|^2$ are equivalent.