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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

June, 2011

MMT-003 : ALGEBRA

Time : 2 hours *Maximum Marks : 50*

Note : *Question No.* ***1*** *is* ***compulsory****. Also do* ***any four*** *questions from Q. No.* ***2*** *to Q. No.* ***6****. Calculators are* ***not*** *allowed.*

1. Which of the following statements are true and which are false ? Give reasons for your answer. **10**

(a) A group of order 15 has a unique subgroup of order 5.

(b) If $ax \equiv bx \pmod{n}$, then $a \equiv b \pmod{n}$ where $a, b, n \in \mathbb{Z}$.

(c) If G is a free abelian group of rank m and if H is a proper subgroup of G of rank n, then $n < m$.

(d) A group of order 36 can have an irreducible representation of degree 6.

(e) If $\alpha = e^{\frac{2\pi i}{7}}$ and $\beta = e^{\frac{2\pi i}{5}}$, $\beta \in Q(\alpha)$.

2. (a) Let $G = GL_n(\mathbf{R})$ operate on the set $S = \mathbf{R}^n$ by left multiplication. Describe the decomposition of S into orbits under this operation. 2

(b) Prove that $\frac{\mathbf{H}}{\{\pm 1\}}$ is isomorphic to the Klein 4-group, where $\mathbf{H}$ is the group of quaternions. Use this fact to obtain all the 1-dimensional representations of $\mathbf{H}$. 6

(c) Let α be a real cube root of 2. Obtain the monic irreducible polynomial satisfied by $1 + \alpha$. 2

3. (a) Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be disjoint cycles of length $m_1, m_2, \ldots, m_k$, respectively, and $\sigma = \sigma_1, \sigma_2 \ldots \sigma_k$. Show that the order of σ is l cm $(m_1, m_2, \ldots, m_k)$. 5

(b) Prove that the subgroup SO_2 of SU_2 is conjugate to the subgroup 5

$$T = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in SU_2 \right\}.$$

4. (a) Prove that every characteristic subgroup is normal. 2

(b) Consider the incomplete character table for a group given below : 8

	(1) 1	(1) a	(2) b	(2) c	(2) d
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	-1	-1
χ_4	2	-2	0	0	0

All the conjugacy classes are there.

(i) What is the order of the group ?

(ii) How many characters are missing ?

(iii) Find the missing character and complete the table.

(iv) Find the order of the Kemel of the missing character.

5. (a) Evaluate the legendre symbol $\left(\frac{7}{61}\right)$ using the quadratic reciprocity law. 3

(b) Give an example of an automaton and justify why it is an automaton. 3

(c) Let K be a field generated over a field F by two elements α, β of relatively prime degrees m, n respectively. Prove that $[K : F] = mn$. 4

6. (a) (i) Show that there is a unique irreducible polynomial of degree 2 over $\mathbb{F}_2$ and find the polynomial. 4

(ii) Let α be a root of the irreducible polynomial. Show that $\sigma(\alpha) = \alpha H$ is a field automorphism.

(b) If F is a finite field, show that there is always an irreducible polynomial of the form $x^3 - x - a$, where $a \in F$ 3

(c) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \in GL_2(\mathbb{F}_5)$. Determine the order of its conjugacy class in $GL_2(\mathbb{F}_5)$. 3