00788

MASTER'S PROGRAMME IN MATHEMATICS WITH APPLICATIONS TO COMPUTER SCIENCE M.Sc. (MACS)

Term-End Examination

June, 2010

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Attempt **five** questions in all. Question number **1** is **compulsory**. Do **any four** questions out of questions 2 to 7. No calculators are allowed.*

1. Are the following statements ***true*** or ***false*** ? Justify your answer with the help of a short proof or a counter example. **5x2=10**

(a) Every subspace of a Banach space is Banach.

(b) On a normed space X, the norm function $\|.\| : X \to \mathbf{R}$ is linear.

(c) If T_1 and T_2 are positive operators on a Hilbert space H, then $T_1 + T_2$ is a positive operator on H.

(d) A closed map on a normed space need not be an open map.

(e) Every finite dimensional normal space is reflexive.

2. (a) Let X be a normed space. Show that the following conditions are equivalent. 7

(i) X is finite dimensional.

(ii) Every closed and bounded subset of X is compact.

(iii) The subset $\{x \epsilon X : ||x|| \leq 1\}$ of X is compact.

(b) Define the eigen spectrum $\sigma_e(A)$ of a bounded linear operator A on a Hilbert space H. Give an example of an Hilbert space H and an operator A on H such that $\sigma_e(A)$ is empty. 3

3. (a) State Riesz representation theorem. Let $H = C^3$ and let $f : H \to C$ be defined by $f((x(1), x(2), x(3))) = x(1) - i\, x(2)$. Find a $y \epsilon H$ that represents f. 4

(b) Show that if a Banach space X is reflexive then its dual X' is also reflexive. 3

(c) Let X be the normed space C [0, 1] with sup norm. 3

For f, g ϵ X given by

$f(x) = 2x - 3$, $g(x) = 2x^2$ for all $x \epsilon [0, 1]$, find $\|f\|$, $\|g\|$. What is the distance between f and g ?

4. (a) State closed graph theorem and use it to prove open mapping theorem. 5

(b) Let A be a normal operator on a Hilbert space X. Show that $\sigma_a(A) \subset \sigma(A)$ where $\sigma_a(A)$ denotes the approximate eigen spectrum of A and $\sigma(A)$ denotes the spectrum of A. 3

(c) Let X be a normed space and Y be a proper subspace of X. Show that the interior Y° of Y is empty. 2

5. (a) Let $\{u_1, u_2, ...\}$ be a countable orthonormal set in an inner product space X. Show that 5

(i) For any $n \epsilon \mathbb{N}$, $\|u_1 + u_2 + ... + u_n\|^2 = n$.

(ii) $\sum_{n=1}^{\infty} |(x, u_n)|^2 \leq \|x\|^2$.

(b) Let X be a normed space and Y be a subspace of X. Show that for every $g \epsilon Y'$, there is at least one Hahn Banach extension of g. 5

6. (a) Let X, Y be normed spaces and suppose BL (X, Y) and CL (X, Y) denote, respectively, the space of bounded linear operators from X to Y and the space of compact linear operators from X to Y. Show that CL (X, Y) is a linear subspace of BL (X, Y). Also, show that if Y is a Banach space, 5

$Fn \epsilon CL (X, Y)$, $F \epsilon BL (X, Y)$ and $\|Fn - F\| \to 0$, then $F \epsilon CL (X, Y)$.

(b) State uniform boundedness principle. 5

Let for each $n \in \mathbf{N}$,

$$x_n(t) = \begin{cases} n^2 t, \text{ if } 0 \le t \le \dfrac{1}{n} \\ \dfrac{1}{t}, \text{ if } \dfrac{1}{n} < t \le 1 \end{cases}$$

Then show that the set $\{x_n : n \in \mathbf{N}\}$ is bounded at each $t \in [0, 1]$, but not uniformly bounded on $[0, 1]$.

7. (a) Define the space NBV ([a, b]). For a fixed $y \in$ NBV ([a, b]) define, $f_y : C[a, b] \to K$, by 3

$f_y(x) = \int_a^b x \, dy$, $x \in C[a, b]$, then show that $f_y \in C[a, b]'$, i.e. f_y is a bounded linear functional on C [a, b].

(b) Give one example of each of the following : 4

(i) A self-adjoint operator as l^2.

(ii) A separable space X whose dual X' is not separable.

(c) Let H be a Hilbert space and G be a non-empty closed subspace of H. Show that $H = G + G^{\perp}$. 3