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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

December, 2010

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Question number 1 is **compulsory**. Do **any four** questions out of questions 2 to 7. Calculators are **not** allowed.*

1. Are the following statements ***true*** or ***false***? Justify your answer with the help of a short proof or a counter example. **5x2=10**

(a) Every finite dimensional normed space is a Banach space.

(b) A continuous map on a normed space is a closed map.

(c) If A and B are unitary operators on a Hilbert space H, then the operator AB is unitary.

(d) C_{00} is a closed subspace of ℓ^{∞}.

(e) The dual of a finite dimensional space is finite dimensional.

2. (a) State Hahn - Banach Extension theorem. Use the theorem to prove the following result : 4

Let X be a normed space over K and $a \in X$ be such that $a \neq 0$. Then there exists an $f \in x'$ such that $f(a) = \|a\|, \|f\| = 1$ and $\|a\| = \text{Sup}\ \{|f(a)| : f \in x',\ \|f\| \leq 1\}|$.

(b) Show that a sub space Y of a Hilbert space is dense if and only if $Y^{\perp} = \{0\}$. 4

(c) Consider $X = K^2$ with two norms given by 2

$$\|x\|_1 = max\{|x_1|, |x_2|\}$$

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$$

Let $T : (X, \|.\|_1) \to (X, \|\cdot\|_2)$ be the linear operator defined by

$T(x_1, x_2) = (x_1 + x_2, -x_1 + x_2)$.

Calculate $\|T\|$.

3. (a) Define an orthonormal set in an inner product space. Check whether $\{u_n\}$ where $u_n = (0,0,..0,1,0,0,...)$, 1 occurs at the n^{th} place, is an orthonormal set in ℓ^2. 3

(b) Let X be a normed space and $P: X \to X$ be a projection. Show that P is closed if and only if the sub spaces $R(P)$ and $Z(P)$ are closed in X. Also show that P in continuous if X is a Banach space. 5

(c) Show that in a normed space the closure of an open ball is a closed ball. 2

4. (a) Let $X = L^p[0,1]$ with $\|\cdot\|_p$ given by **5**

$$\| f \|_p = \left[\int_0^1 | f(t) |^p \, dt . \right]^{1/p}, f \in X.$$

Here $1 \leq p < \infty$.

Let f and g be functions given by

$f(t) = t,\ g(t) = 1-t,\ \forall t \in [0,1]$

compute the following :

(i) $\| f \|_p$ (ii) $\| g \|_p$

(iii) $\| f+g \|_p$ and (iv) $\| f-g \|_p$.

Also show that the following relation holds for $p=2$.

$$\| f+g \|_2^2 + \| f-g \|_2^2 = 2 \left(\| f \|_2^2 + \| g \|_2^2 \right).$$

(b) Let p, q be such that $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. **3**

Let $f \in L^q [0,1]$. Show that $f \in L^{p^*} [0,1]$, i,e. f defines a bounded linear functional on $L^p [0,1]$

(c) Let X be a normed space. Show that every bounded linear functional on X is a compact operator from X to K. **2**

5. (a) Let X and Y be normed spaces. Then show that the following hold : 6

(i) If Y is a Banach space, then $BL(X,Y)$ is a Banach space.

(ii) If X is a Banach space, then the quotient space X/Y is a Banach space.

(b) Let B be a convex subset of a normed space X. Show that the interior B^o and the closure $\overline{B}$ are convex subsets of X. If $B^o \neq \phi$, then show that $\overline{B} = \overline{B^o}$. 4

6. (a) Let $H = \ell^2$. Let A be the operator on H given by : 4

$$A(\alpha_1, \alpha_2, \ldots\ldots\ldots) = (0, \alpha_1, \alpha_2, \ldots\ldots\ldots).$$

Find A^*. Is A normal ? Justify.

(b) Let X and Y be normed spaces. 6

Let $F \in BL(X,Y)$ and $F' \in BL(Y', X')$

be given by :

For $y' \in Y'$ and $x \in X$,

$F'(y)'(x) = y'(F(x))$.

Then show that

(i) $Z(F') = \{y' \in Y' : y'(y) = 0$ for all $y \in R(F)\}$.

(ii) $Z(F) = \{x \in X : x'(x) = 0$ for all $x' \in R(F')\}$

(iii) F' is one - to - one if and only if $R(F)$ is dense in Y.

7\. (a) Define a reflexive normed space. Show that a closed subspace of a reflexive space is reflexive. Use this result to show that the normed space ℓ^∞ is not reflexive. 5

(b) Let $\{ u_1, u_2 \ldots\ldots \}$ be a countable orthonormal set in an inner product space X and $x \in X$. Then show that : 5

$$\sum_n |\langle x, u_n \rangle|^2 \leq \|x\|^2 .$$