00219

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

December, 2010

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Question no. 1 is **compulsory**. Do **any four** questions out of question nos. 2 to 7. Calculators are **not** allowed.*

1. State, whether the following statements are *True* or *False*. Give reasons for your answer : **5x2=10**

   (a) For $X = [0, 1]$ with the standard metric,

   $$B\left(0, \frac{1}{2}\right) = \left]0, \frac{1}{2}\right[.$$

   (b) Continuous image of a Cauchy Sequence is always a Cauchy Sequence.

   (c) The outer measure $m^*(z)$ for the set of integers, z, is zero.

   (d) Every closed and bounded subset of a metric space is compact.

   (e) If a set E has finite measure, then $L^2(F) \subset L^1(E)$.

2. (a) Let (x, d) be a metric space. Show that the function $D : X X \to R$ defined by 3

$$D(x, y) = \frac{4d(x, y)}{1+4d(x, y)}$$ is a metric on X.

(b) Let $F : R^4 \to R^3$ be defined by 3

$F(x, y, z, w) = (x^2+y^2+zw^2, x^2y, xyz)$

Find : $F'(Q)$ at $Q=(1, 2, -1, -2)$.

(c) Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets and let $m(E_1)$ be finite. 4

Then show that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n)$.

Verify this result for $\{E_n\}$ when $E_n = \chi[0, \frac{1}{n}]$.

3. (a) Let X be a metric space. Show that if a sequence $\{x_n\}$ converges to x in X, then every subsequence of $\{x_n\}$ also converges to x. 5

Is the converse of the above result true ? Justify.

(b) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E, such that $|f_n| \leq g$ a.e on E for all $n \geq 1$, where g is integrable over E. If $f(x) = \lim_{n \to \infty} f_n(x)$ a.e, then prove that f is integrable over E and 5

$$\int_E f\, dm = \lim_{n \to \infty} \int_E f_n\, dm$$

4. (a) State inverse function theorem. Apply inverse function theorem to check the local invertibility of the function $f : \mathbf{R}^3 \to \mathbf{R}^3$, defined by 5

$$f(x, y, z) = (x+y+z, e^x \cos z, e^x \sin z).$$

(b) Find the Fourier Sine Series of the function $f(x) = x$ in $-\pi < x < \pi$. 3

(c) Show that every finite subset of a metric space is totally bounded. 2

5. (a) (i) Define a connected metric space. 6

(ii) Prove that a metric space X is connected if, then every continuous function, on X to the discrete metric space $\{-1, 1\}$ is a constant function.

(iii) Show that the metric space Q of all rational numbers with the usual metric $d(x, y) = |x - y|$, is not connected.

(b) Let f and g be, the functions given by 4

$$f(t) = \begin{cases} \sqrt{t}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \leq 0 \text{ or } t \geq 1 \end{cases}$$

$$g(t) = \begin{cases} \sqrt{1-t}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \leq 0 \text{ or } t \geq 1 \end{cases}$$

Find the convolution $f * g$ of f and g.

6. (a) The impulse response of an LTI system is $h(t) = e^{-2t} u(t)$. Find the system response to the input function 5

$$f(t) = \sum_{k=-2}^{2} \left(\frac{1}{2}\right)^k e^{i3kt}.$$

(b) Find the interior, closure and boundary of the set 5

$B = \{ (x, y) \in R^2 : y = 0 \}$ in R^2.

7. (a) Check the function $f : R^3 \rightarrow R$ defined by, 4

$$f(x, y, z) = x^2y + y^2z + z^2 - 2x$$

for local extrema.

(b) Explain the following terms with an example : 6

(i) Stable system

(ii) Causal system

(iii) Memory - less system