M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00098

June, 2015

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *of the remaining questions.*

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1. State whether the following statements are *true* or *false*. Give a brief justification, with a short proof or a counter example, to support your answer. *5×2=10*

(a) The image of a convex set under a real linear functional is an interval.

(b) In a Banach space, every absolutely convergent series is convergent.

(c) If X is a normed linear space, $x, y \in X$ and $f(x) = f(y)$ for every $f \in X'$, then $x = y$.

(d) If M is a linear subspace of a Hilbert space H and if $M^{\perp} = (0)$, then $M = H$.

(e) A bounded linear operator A on an infinite-dimensional Hilbert space cannot be compact, if $A^3 = I$.

2. (a) Prove that a normed linear space is finite-dimensional, if the unit ball is compact. 3

(b) If a Hilbert space H has a countably infinite orthonormal basis, then show that H is linearly isometric to l^2. 4

(c) Prove or disprove : Every bounded linear operator on a complex Hilbert space has an eigenvalue. 3

3. (a) If (a_n) is a sequence such that $(a_n b_n)$ converges to zero for every sequence (b_n) converging to zero, then prove that $(a_n) \in l^\infty$. 4

(b) Let X be a Banach space, and $A_1B \in BL(X)$. Show that if B is compact, the spectra of A and A + B are the same, except for eigenvalues. 4

(c) Find $\{(1, 1, 0), (1, 0, -1)\}^\perp$ in $\mathbf{R}^3$. 2

4. (a) Give an example of a compact self-adjoint operator on l^2, with justification. 3

(b) Give an example of a non-zero bounded linear operator that is not open, with justification. 2

(c) Prove that the dual of c_0 is linearly isometric to l^1. 5

5. (a) If f is a linear functional on a normed linear space whose zero space Z(f) is closed, prove that f is continuous. 2

(b) Let X be a Banach space with norm $\| \cdot \|$. Let $l^{\infty}(X)$ denote the linear space of bounded sequences in X. Show that $l^{\infty}(X)$ is a Banach space with norm $\| (x_n) \|_{\infty} := \sup_n \| x_n \|$. 5

(c) Let H be a Hilbert space and let u, v $\in$ H. Define $Ax = \langle x, u \rangle v$, $x \in H$. Prove that A is a linear, compact operator. 3

6. (a) Compute the norm of the linear map

$(\mathbf{C}^2, \| \cdot \|_1) \to (\mathbf{C}^2, \| \cdot \|_{\infty}), (z_1, z_2) \to (z_1, z_2)$. 2

(b) Prove that a bounded linear operator A on a Hilbert space H is normal iff $\| A^* x \| = \| Ax \| \ \forall x \in H$. 3

(c) Let X be the real normed linear space $(\mathbf{C}^4, \| \cdot \|)$ and let $Y = \mathbf{R}^2$ be its subspace. Prove that Y is a closed subspace of X and that X / Y is a normed linear space. Further, give 2 distinct elements of X / Y with positive norm. 5