M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00468

Term-End Examination

June, 2015

MMT-003 : ALGEBRA

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. 1 is **compulsory**. Answer any **four** questions from questions no. **2** to **6**. Use of calculator is **not** allowed.*

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1. State with reasons, which of the following statements are *True* and which are *False* : *5×2=10*

(a) The system of congruences

$$2x \equiv 3 \pmod 2$$

$$3x \equiv 2 \pmod 3$$

has a unique solution modulo 6.

(b) The splitting field of a polynomial f(x) over a finite field F is finite.

(c) If a finite group G acts on a finite set S, then $G_{gs} = G_s$ for all $s \in X$ and $g \in G$.

(d) If G is a finite group of order 16, then G cannot have an irreducible representation of degree greater than two.

(e) If p, q are primes and p is a quadratic residue modulo q, then q is a quadratic residue modulo p.

2. (a) Show that the finite field $\mathbf{F}_{16}$ contains a subfield of order 4, but not of order 8. *2*

(b) What are the degrees of splitting fields of the polynomials

(i) $f(x) = x^4 - 1$

(ii) $g(x) = x^4 + 1$

over $\mathbf{Q}$? *3*

(c) State and prove Chinese Remainder Theorem. *5*

3. (a) How many Sylow 11-subgroups are there for a group G of order 484 ? *3*

(b) How many irreducible representations are there for the cyclic group $C_3 = \{1, a, a^2\}$? *2*

(c) Let G be a group and $\rho : G \to GL(n, \mathbf{R})$ be a faithful representation. Suppose $\rho(g)$ is a diagonal matrix $\forall g \in G$. Show that G is abelian. *2*

(d) Is there a field of order 9 ? If yes, list all the elements of the field. *3*

4. (a) Show that $h(e^{i\theta}) = \begin{bmatrix} e^{i\theta} & e^{i2\theta} - e^{i\theta} \\ 0 & e^{i2\theta} \end{bmatrix}$

is a representation of the circle group $S^1 \cong SO(2)$. *3*

(b) Let $S = \left\{\overline{1}, \overline{2}, ..., \overline{\frac{p-1}{2}}\right\}$ be the subset of residue classes modulo p. For $s \in S$ and $a \in \mathbf{Z}$, $p \times a$, let $e_s(a)$ be such that $sa = e_s(a)\, a$. Show that $\left(\frac{a}{p}\right) = \prod_{s \in S} e_s(a)$. *5*

(c) Show that the number of irreducible representations of a finite group G does not exceed the number of elements in the group. *2*

5. (a) Find a generator for the cyclic multiplicative group $\mathbf{F}_7^*$ (of the field $\mathbf{F}_7$), $\mathbf{F}_7^* = \mathbf{F}_7 \setminus \{0\}$. *2*

(b) Let F be a field of characteristic $\neq 2$. Let α and β be roots of $X^2 - a \in F(X)$ and $X^2 - b \in F(X)$, respectively. Show that $(\alpha + \beta)$ is a root of

$$X^4 - 2\,(a + b)\,X^2 + (a - b)^2.$$ *5*

(c) Let $G = \{1, a, a^2\}$ be the cyclic group of order 3 and let ρ be a representation of G given by $\rho(a) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Check whether ρ is irreducible. *3*

6. (a) Suppose K is an extension field of F and the degree $[K : F] = 17$. Show that if $a \in K \setminus F$, then the degree of a is 17 over F. *2*

(b) Let $\sigma \in A_n$. If $Z(\sigma)$ has only even permutations, then show that partition of n corresponding to σ are odd and distinct. *5*

(c) Let **F** be a finite field with q elements. Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbf{F})$. Find N(a) and its order. *3*