No. of Printed Pages : 10 **MMT–008**

M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. SC. (MACS)]

Term-End Examination

June, 2024

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Note : *(i) Question No. 8 is compulsory. Attempt any* ***six*** *questions from question nos.* ***1*** *to* ***7****.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

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1. (a) A study about a population showed that the mobility of population of a state to a village, town and city is in the following percentages :

From	**To**		
	Village	**Town**	**City**
Village	60	25	15
Town	10	70	20
City	10	30	60

P. T. O.

What will be the proportion of population in village, town and city after one year and two years, given that the present proportion of the population in the village, town and city are respectively 0.50, 0.40 and 0.10 ? 6

(b) In a certain state, 58 landfills are classified according to their concentration of three hazardous chemicals : arsenic, barium and mercury. Suppose that the concentration of each one of the three chemicals is characterized as either high or low. If a landfill is chosen at random from among 58 landfills, given the following configuration :

	Barium			
	High Mercury		Low Mercury	
Arsenic	High	Low	High	Low
High	1	3	5	9
Low	4	8	10	18

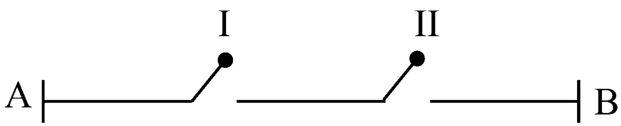
Find the probability that it has :

(i) high concentration of Barium,

(ii) high concentration of mercury and low concentration of both Arsenic and Barium and

(iii) high concentration of any *one* of the chemicals and low concentration of the other two. 9

2. (a) Consider the following system consisting of two switches I and II between two points A and B :



A signal is sent from the point A to point B and is received at B if both the switches I and II are closed. It is assumed that the probabilities of I and II being closed are 0.8 and 0.6 respectively and that P [II is closed | I is closed] = P[II is closed].

Find :

(i) The probability that signal is received at B,

(ii) The conditional probability that switch I was open, given that the signal was not received at B,

(iii) The conditional probability that switch II was open, given that the signal was not received at B. 8

(b) For the (M/M/K) : (∞ / FIF 0) queuing model with arrival rate λ and service rate per service channel μ, obtain the steady-state probability of n customers in the system, P_n. Hence or otherwise, find the probability that an arrival has to wait. 7

3. (a) The certain item is manufactured by three factories, say 1, 2 and 3. It is known that 1 turns out twice as many items as 2 and that 2 and 3 turn out the same number of items (during a specified production period). It is also known that 2 percent of the items produced by 1 and 2 are defective while 4 percent of those manufactured by 3 are defective. All the items produced are put into one stock pile and then one item is chosen at random. The chosen item was found defective. What is the probability that it was produced in factory 1 ? 7

(b) The joint distribution of the random variables X and Y is given by :

y \ x	-1	0	1
-1	α	β	α
0	β	0	β
1	α	β	α

where $\alpha, \beta > 0$ with $\alpha + \beta = 1/4$.

(i) Derive the marginal distribution of X and Y.

(ii) Calculate the E(X), E(Y) and E(XY).

(iii) Show that Cov(X, Y) = 0.

(iv) Show that the variables X and Y are dependent. 8

4. (a) Let $\{X_n; n \geq 0\}$ be a Markov chain with four states; 1, 2, 3, 4 and the following transition probability matrix :

	1	2	3	4
1	0	0	1	0
2	1	0	0	0
3	1/2	1/2	0	0
4	1/3	1/3	1/3	0

(i) Find the probability $P[X_3 = 3, X_2 = 1 \mid X_1 = 2]$.

(ii) Classify the states of the given Markov chain. 8

(b) It is claimed that the function $F_{X,Y} = \frac{1}{16} xy(x + y), \quad 0 \le x \le 2, 0 \le y \le 2$ is the joint distribution function of the random variables X and Y. Then :

(i) determine the corresponding joint probability density function $f_{X,Y}$ and

(ii) Calculate the probability $P(0 \le X \le 1, 1 \le Y \le 2)$. 7

5. (a) In an investigation related to a specific type of scores of men and women aged 65 to 70, the mean verbal and performance scores for 101 subjects were found to be :

$$\bar{X} = \begin{bmatrix} 55.24 \\ 34.97 \end{bmatrix}$$

The sample covariance matrix of the scores was

$$S = \begin{bmatrix} 210.54 & 126.99 \\ 126.99 & 119.68 \end{bmatrix}$$

In order to test the null hypothesis that observations came from a population with mean vector $\mu_0 = \begin{bmatrix} 60 \\ 50 \end{bmatrix}$, apply a suitable test statistic. You may consider $\alpha = 0.01$ for the test.

[You may like to use the following values $F_{0.01,2,99} = 4.98$, $F_{0.01,2,20} = 5.85$] 6

(b) What is the purpose of principal component analysis ? Given the covariance matrix of order 2×2, explain how would you extract the principal components. Also, explain how would you find the proportion of total population variance for all the principal components. 9

6. (a) Distinguish between 'Age Replacement' and 'Block Replacement' policies.

Let the lifetimes Y_1, Y_2, ..., are independently and identically distributed random variables and follows negative exponential distribution with parameter 5.

If lifetimes T > 0 and age replacement policy is to be employed, Then :

(i) find the mean renewal time and

(ii) find the long-run average cost per unit time, given the costs $C_1 = 4$ and $C_2 = 6$ units of money. 8

(b) In order to fit the regression line $y = \beta_1 + \beta_2 x$ on a data set consisting of 34 pairs of values (z, y), the least square estimates of β_1 and β_2 are to be computed. From the data set, the following values are obtained :

$$\sum_i x_i = 100.73, \qquad \sum_i y_i = 16,703,$$

$$\sum_i x_i^2 = 304.7885, \quad \sum_i x_i y_i = 50,006.47.$$

Obtain the fitted regression line. 7

7. (a) What is branching process ? Give *two* real examples of branching process.

If $P(s)$ and $P_n(s)$ respectively is the probability generating function (*pgf*) of the i.i.d. random variables $\{\xi_r\}$ and the random variables $\{X_n\}$, where

$$X_{n+1} = \sum_{r=1}^{X_n} \xi_r .$$

then show that :

$$P_n(s) = P_{n-1}(P(s))$$

$$\text{and } P_n(s) = P(P_{n-1}(s)).$$ 8

(b) A community has two police cars, which operate independently of one another. The probability that a specific car will be available when needed is 0.99.

(i) What is the probability that neither car is available when needed ?

(ii) What is the probability that a car is available when needed ? 7

8. State whether the following statements are true *or* false. Justify your answer with a short proof or a counter example : 10

(i) Although in one-step transition probability matrix, P, of a Markov chain the sum of each row must necessarily be unity but in higher order transition probability matrices, $P^{(j)}$; $j = 2, 3, 4, ...$. This rule is not necessary.

(ii) For the joint pdf of random variables (X, Y) given by :

$$f(x, y) = x^2 + \frac{xy}{3},\ 0 \le x \le 1,\ 0 \le y \le 2$$

$$P[X + Y \ge] = \frac{65}{72}.$$

(iii) Let the r.v. X follows the negative exponential distribution with parameter $\lambda = \frac{1}{3}$. Then

$$P[X > 4 \mid X > 3] = P[X > 1] = e^{-\frac{1}{3}}.$$

(iv) One of the examples of non-Poisson queuing system is the (M/G/1) : (∞/F1F0) queuing model.

(v) If $X_1, X_2, \ldots, X_n$ be a random sample from $N_p(\mu, \Sigma)$, then maximum likelihood estimators of μ and Σ are :

$$\hat{\mu} = \bar{X} \text{ and } \hat{\Sigma} = \sum_{i=1}^{n} (X_i - \bar{X})'(X_i - \bar{X}).$$