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M.Sc. MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE

[M.Sc. (MACS)]

Term-End Examination

June, 2024

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *Questions number **6** is compulsory. Attempt any **four** of the remaining questions.*

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1. (a) Suppose $\{e_n\}$ is an orthonormal basis for a Hilbert space H and f is a bounded linear functional on H. Show that $f(x) = \langle x, y\rangle$ for all $x \in \text{H}$, where $y = \sum \overline{f(e_n)}\, e_n$ and $\|f\|^2 = \sum |f(e_n)|^2$. 4

P.T.O.

(b) Prove that a normed space X is complete if there is a complete subspace M such that X/M is complete. 3

(c) Find a sequence (A_n) of bounded operators on l^2 such that $\|\text{A}_n\| = 1$ for all n and $\lim_{n\to\infty} \text{A}_n x = 0$ for all $x \in l^2$. 3

2. (a) State the principle of uniform boundedness. Use it to prove that a sequence (A_n) of bounded operators on a Hilbert space H is bounded if $\{\langle A_n x, y\rangle\}$ is a bounded sequence for all $x, y \in \text{H}$. 3

(b) Let $\text{M} = \{x \in l^4 : x_{2n-1} = 0 \text{ for all } n\}$. Show that l^4 and M are homeomorphic. 4

(c) Find an orthonormal basis for span $\{(1, 0, 1), (0, 1, 1)\}$ in R^3. 3

3. (a) Let X be a real normed linear space and let x_0 be a non-zero vector in X. Using Hahn-Banach theorem show that **J** a linear functional f_0 on X. Such that 4

$$f_0(x_0) = \|x_0\| \text{ and } \|f_0\| = 1.$$

(b) Let u, v be non-zero vectors in a Hilbert space H and $Ax = \langle x,n \rangle v + \langle x, v) \rangle u$ for $x \in H$. If $u \perp v$, show that A is a bounded linear operator with $\| A \| = \| u \| \, \| v \|$. Compute A^*. 6

4. (a) Find a normed space X and $x, y \in x$. Such that $\| x \| = \| y \| = 1$ and $\| x + y \| = 2$. 2

(b) On [0, 1] show that $Af(x) = \int_0^x f(t)dt$ defines a bounded linear map. Also, calculate $\| A \|$. 4

(c) If A is a selfadjoint operator on a Hilbert space prove that $\sum \frac{(iA)^n}{n!}$ converges in norm. 4

5. (a) If $\|.\|_1$, $\|.\|_2$ are norms on a vector space X, show that $\| x \| = \max (\| x \|_1, \| x \|_2)$ defines a norm. 3

(b) Prove that a non-zero linear functional f on a normed space is continuous if and only if $Z(f_0)$ is not dense. 4

(c) Let M be a linear subspace of a normed space X and $i : M \to X$ be the inclusion map $i(y) = y$. Calculate i^*, the conjugate map of i. 3

6. State, with justification, whether the following statements are True or False : $5 \times 2 = 10$

(a) If A is a bounded operator and $\sigma(A) = \{0\}$, then $A = 0$.

(b) In l^2, $\{e_n : n = 1, 2, ...\}^1 = (0)$.

(c) $f \to f(0)$ is a bounded linear functional on $(C[-1, 1], \|.\|_1)$.

(d) If M is a proper subspace of a normed space X, then X, M is dense in X.

(e) Let $\{x_n\}$ be a sequence in a normed linear space X. If $\| x_m \| \to \| x \|$ for some $x \in X$, then $x_m \to x$.

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