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M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

June, 2023

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Weightage : 50%

Note : *(i) Question No. 8 is **compulsory**. Attempt any **six** questions from Question Nos. **1** to **7**.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

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1. (a) Let $Y \sim N_3(\mu, \Sigma)$, where $\mu = [6, 0, 8]$ and $\Sigma = \begin{pmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{pmatrix}$. Obtain the distribution of $X = CY$, where $C = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. 6

P. T. O.

(b) Let X_1 and X_2 be two random variables with joint probability distribution given as : 9

X_2 \ X_1	1	2	3
1	0.08	0.17	0
2	0	0.11	0.20
3	0.06	0.25	0.13

Find :

(i) The variance, covariance matrix and mean for vector $X = (X_1, X_2)$.

(ii) Conditional variance $V(X_2 \mid X_1 = 2)$.

2. (a) Determine the principal components Y_1, Y_2 and Y_3 for the covariance matrix :

$$\Sigma = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Also calculate the proportion of total population variance for all the principal components and interpret the result. 9

(b) Let $\{X_n ; n \geq 0\}$ is a branching process, then for $r = 0, n = 0, 1, 2, \ldots\ldots\ldots,$ show that $E\left[X_{n+r} \mid X_n\right] = X_n m^r$. 6

3. (a) Let $\{X_n ; n \geq 0\}$ be a Markov chain having state space S = {1, 2, 3, 4} and transition probability matrix : 8

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(i) Classify the states of a given Markov chain i. e. recurrent, transient, periodic and mean recurrence time.

(ii) Whether the chain is irreducible ?

(iii) Does the limiting probability vector exist ? If yes, find the limiting probability vector. If not, give reasons.

(b) The tooth care hospital provides free dental service to the patients on every Saturday morning. There are 3 dentists on duty who are equally qualified and experienced. It takes on an average 20 minutes for a patient to get a treatment and the actual time taken is known to vary exponentially around this average. The patients arrive according to Poisson distribution with an average of 6 per hour. The administrative officer of hospital wants to investigate the following : 7

(i) The expected number of patients waiting in the queue.

(ii) Average time that a patient spends at the clinic.

(iii) Average percentage idle time for each of the dentists.

(iv) The fraction of time at least one dentist is idle.

(v) Traffic intensity of the system.

4. (a) The following data show the number of ever born children and dead children to a number of couples belonging to low and medium income group :

Low Income Group	No. of Children
6	2
4	0
5	1
1	0
2	0
4	0
7	2
8	1
1	0
12	2
2	0
5	0
3	0
10	3

Middle Income Group	No. of Children
2	0
3	0
2	0
3	0
7	2
8	1
4	0
5	0
6	0
7	0
6	1
8	3
10	2
5	1
4	0
3	0
6	1
4	1
5	0
2	0

Test the hypothesis that average ever born children and dead children to low and medium status couples are equal, assuming that the no. of ever born and dead children follow bivariate normal population with covariance matrices are equal but unknown. 9

(b) A doctor is to visit the patient and from past experience, it is known that the probability that he will come by train, bus or scooter are respectively $\frac{3}{10}, \frac{1}{5}$ and $\frac{1}{10}$, the probability that he will use some other means of transport being therefore $\frac{2}{5}$. If he comes by train, the probability that he will be late is $\frac{1}{4}$; if by bus $\frac{1}{3}$; and if by scooter $\frac{1}{12}$. If he uses some other means of transport it can be assumed that he will not late. What is the probability that he has come by bus : 6

(i) if he is late.

(ii) if he is not late ?

5. (a) For $M|M|1$ queuing system with arrival rate λ and service rate μ per unit time, where $\lambda < \mu$, determine the steady state probability P_n if not more than N customers are allowed in queuing system. Also find an expression for average queue length. 8

(b) If (X, Y) follows bivariate normal distribution with :

$$\mu_X = 3,$$
$$\mu_Y = 1,$$
$$\sigma_x^2 = 16$$
$$\sigma_y^2 = 25$$

and $\rho = \frac{3}{5}$.

Find : 7

(i) $P\left(3 < Y < 8 \mid X = 7\right)$; and

(ii) $P(-3 < X < 3 \mid Y = -4)$

6. (a) Find the regression equation :

$$Y = b_0 + b_1 X_1 + b_2 X_2$$

for the data given below : 8

Y	X_1	X_2
12	8	1
6	3	2
3	5	2
22	17	5
4	9	6
10	16	8

Also find the multiple correlation co-efficient and mean square error.

(b) For the Yule-Furry process starting with only one individual at $t = 0$, find the density function of T_n (the time at which the population first reaches the value $n(>1)$. Find E (T_n). 7

7. (a) Suppose the life times X_1, X_2, ... are i. l. d. uniformly distributed on $[2, 5]$ years and it is assumed that planned replacement take place every 3 years so that a machine is replaced on failure or at the end of 3 years. Calculate : 8

(i) Long-term rate of replacement

(ii) Long-term rate of failures

(iii) Long-term rate of planned replacements

(b) Consider a system with 3 components which are arranged in series. The system fails if any or either of the 3 components fail. On failure, a component is replaced instaneously. Suppose each component works independenently and has exponential life time distribution with parameter λ. Let N_t denote the no. of failures for the system in the time interval $[0, t]$. Find the distribution of N_t and also the renewal function. 7

8. State whether the following statements are true or false. Justify your answer with a short proof or a counter-example : 10

   (i) If P is a transition matrix of a Markov chain, then all the coloumn sums of $\lim_{n\to\infty} P^n$ are unity.

   (ii) t-test is used to test the statistical significance of equality of two or more population variance-covariance matrices.

   (iii) If $(X, V) \sim$ Bivariate normal $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho_{xy})$, then the marginal p. d. f.'s of and X and Y are not normal.

   (iv) The renewal function M (t) satisfies the equation :

$$M(t) = F(t) - \int_0^t M(t-x)\, d\,F(x)$$

   (v) If X_1 X_2, X_3 and X_4 are i.i.d. from $N_2(\mu, \Sigma)$, then :

$$\frac{1}{2}(X_1 - 2X_2 + 3X_2 + 4X_4) \sim N_2\left(\frac{1}{2}\mu, \Sigma\right).$$