No. of Printed Pages : 6

MMT-004

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

June, 2023

MMT-004 : REAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No. **1** is compulsory.*

*(ii) Attempt any **four** questions from Q. Nos. 2 to 6.*

(iii) Calculator is not allowed.

(iv) Notations as in the study material.

---

1. State whether the following statements are True *or* False. Give reasons for your answers : 5×2=10

(a) If (X_1, d_1) is any discrete metric space and (X_2, d_2) is any metric space, then any function $f : X_1 \to X_2$ is continuous.

(b) If $\{x_n\}$ is a sequence in a metric space (X, d) such that every subsequence of $\{x_n\}$ is convergent in X, then $\{x_n\}$ itself is convergent in X.

(c) **R** with the Euclidean metric is an example of a non-compact, non-connected space.

(d) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. The existence of the partial derivatives of f at a point in $\mathbf{R}^2$ implies the differentiability of f at that point.

(e) If $E_n = [n, n+1]$, then $m(E_n) = 0$.

2. (a) Suppose d_1 and d_2 are two metrics on a set X. Prove that d_1 and d_2 are equivalent if and only if for any given $x \in X$ and $r > 0$, there exists reals $r_1 > 0, r_2 > 0$ such that : 3

$$B_{d_2}(x, r_1) \subset B_{d_1}(x, r) \subset B_{d_1}(x, r_2) \subset B_{d_2}(x, r)$$

(b) Apply inverse function theorem to check the local invertibility of the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by : 3

$$f(x, y, z) = (2x + 2y + 2z, e^x \cos z, e^x \sin z)$$

(c) Let E be a measurable subset of $\mathbf{R}$ with $m(\mathrm{E}) = \alpha$. Show that $\mathrm{E} + \mathrm{S}$ is also measurable. What is measure of $\mathrm{E} + \mathrm{S}$? 4

3. (a) Let (X, d) be a metric space and $f : \mathrm{X} \to \mathbf{R}$ be a continuous function. Prove that for each $t \in \mathbf{R}$, the set $\{x \in \mathrm{X} : f(x) > t\}$ is open in X. 2

(b) Obtain the Taylor's series expansion upto 2nd derivative for the function : 4

$$f(x, y) = \sin(x + y) \text{ at } \left(\frac{\pi}{2}, 0\right).$$

(c) Let $\{\mathrm{E}_n\}$ be an infinite decreasing sequence of measurable sets with $m(\mathrm{E}_1) < \infty$. Prove that : 4

$$m\left(\bigcap_{n=1}^{\infty} \mathrm{E}_n\right) = \lim_{n \to \infty} m(\mathrm{E}_n)$$

What happens if $m(\mathrm{E}_1) < \infty$? Justify.

4. (a) Prove that $\mathrm{Q} \times \mathrm{Q}$ is not complete. 4

(b) Suppose $f : \mathbf{R} \to \mathbf{R}^2$ is given by $f(t) = (t, t^2)$ and $g : \mathbf{R}^2 \to \mathbf{R}^3$ is given by $g(x, y) = (x^2, xy, y^2 - x^2)$. Compute the derivative of $g \circ f$. 3

(c) Let f be a non-negative measurable function and E be a measurable subset of $\mathbf{R}$. Define $\int_{\mathrm{E}} f \, dm$. 3

Let $f : \mathbf{R} \to \mathbf{R}$ be defined as :

$$\begin{aligned} f(x) &= 1, \quad x \in [2, 3] \\ &= \frac{1}{2}, \quad x \in [4, 5] \\ &= 3, \quad x \in [6, 8] \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

Evaluate $\int_{\mathbf{R}} f \, dm$.

5. (a) Prove that in a metric space, finite union of compact sets is compact whereas arbitrary union of compact sets need not be compact. 4

   (b) Verify the implicit function theorem and verify the same for the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by $f(x, y, z) = x^2 y - e^x + z$ near the point (1, 1). 4

   (c) Define a stable system and give an example. 2

6. (a) Suppose $\{A_\alpha\}_{\alpha \in I}$ be an arbitrary collection of connected sets in a metric space (X, d). If $\bigcap_\alpha A_\alpha \neq \phi$, then prove that $\bigcup_\alpha A_\alpha$ is connected. 3

   (b) Find the critical points of the function $f(x, y, z) = x^2 y + y^2 z + z^2 - 2x$. Classify the critical point. 3

   (c) Define the Fourier transform of a function $f \in L^1(\mathbf{R})$. Prove that the Fourier transform is a continuous function on $\mathbf{R}$. 4