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MMT-007

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

June, 2022

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 hours | *Maximum Marks : 50*

(Weightage : 50%)

Note : *Question no.* ***1*** *is* ***compulsory****. Attempt any* ***four*** *questions out of the remaining questions no. 2 to 7. Use of non-programmable scientific calculator is allowed.*

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1. State whether the following statements are *True* or *False*. Justify your answer with the help of a short proof or a counter example. *5×2=10*

(a) The solution of IVP

$$y' = -|y|, \; y(0) = 1$$

does not exist at any point in the neighbourhood of the origin.

(b) When the heat conduction equation $u_t = u_{xx}$ is approximated by

$$\frac{1}{2k}(u_m^{n+1} - u_m^{n-1}) = \frac{1}{h^2}(u_{m-1}^n - 2u_m^n + u_{m-1}^n),$$

the transaction error of the method is of order $(k^2 + kh^2)$.

(c) For the differential equation $(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$, $x = 1$ is a regular singular point.

(d) $\mathcal{L}\left[\frac{\sinh 2t}{t}\right] = \frac{1}{2} \ln\left(\frac{s+2}{s-2}\right)$.

(e) For solving the IVP

$$y' = f(x, y),\ y(x_0) = y_0,$$

a general multi-step method can be written in the form

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \dots + a_k\, y_{i-k+1} + h(b_0 y'_{i+1} + b_1 y'_i + \dots + b_k y'_{i-k+1}).$$

The method is explicit if $b_0 \neq 0$ and implicit if $b_0 = 0$.

2. (a) Construct Green's function for the boundary value problem *5*

$$\frac{\partial^2 y}{\partial x^2} + 2\frac{\partial y}{\partial x} + 10y = 0,\ 0 < x < \frac{\pi}{2}$$

$$y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0.$$

(b) Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(x, 0) = 0,\ u(0, t) = 0 \text{ and } u(1, t) = t,$$

using implicit Crank–Nicolson method with $h = \frac{1}{2}$ and $k = \frac{1}{8}$. Integrate for two time levels. *5*

3. (a) Find the power series solution, near $x = 0$ of the differential equation

$$9x(1-x)\,y'' - 12y' + 4y = 0.$$ *6*

(b) Expand $f(x) = x^3 - 3x^2 + 2x$ in a series of the form $\sum_{n=0}^{\infty} a_n H_n(x)$, where $H_n(x)$ is the Hermite polynomial of degree n in x. *4*

4. (a) The five-point formula for Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = G(x, y) \text{ is}$$

$$u_{i+1, j} + u_{i-1, j} + u_{i, j+1} + u_{i, j-1} - 4u_{i, j} = h^2 G(x_i, y_j).$$

Using Taylor Series expansions, find the order of this five-point formula. *5*

(b) Solve the initial value problem $y' = -2xy^2$, $y(0) = 1$ with $h = 0{\cdot}2$ on the interval $[0, 0{\cdot}4]$, using the Predictor-Corrector method.

$$P : y_{k+1} = y_k + \frac{h}{2}(3y'_k - y'_{k-1})$$

$$C : y_{k+1} = y_k + \frac{h}{2}(y'_{k-1} + y'_k)$$

Perform two corrector iterations per step. Use the exact solution $y = \frac{1}{1+x^2}$ to obtain the starting value. *5*

5. (a) Using Laplace Transform, solve

$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$ with $u(0, t) = u(2, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = 10 \sin 2\pi x$. *5*

(b) Find the solution to the initial boundary value problem, subject to given initial and boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \begin{bmatrix} 2x & \text{for} & x & \in & \left[0, \frac{1}{2}\right] \\ -2x & \text{for} & x & \in & \left[\frac{1}{2}, 1\right] \end{bmatrix}$$

$$u(0, t) = 0 = u(1, t),$$

using Schmidt method with $\lambda = \frac{1}{6}$ and $h = 0{\cdot}2$. *5*

6. (a) Find the solution of the boundary value problem

$$u_{xx} + u_{yy} = x^2 + y^2$$

$$u(x, y) = x^2 - y^2 \text{ on the boundary}$$

$$0 \leq x \leq 4, \; 0 \leq y \leq 4, \; 0 \leq x + y \leq 4,$$

using the five-point formula. Assume h = 1. *6*

(b) If $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1, \end{cases}$

show that

$$f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_n(x) + \ldots,$$

where $P_n(x)$ is a Legendre polynomial of degree n. *4*

7. (a) Using Runge-Kutta method of 4^{th} order, find y(0·8), take h = 0·1, correct to three decimal places, if

$$\frac{dy}{dx} = y - x^2, \; y(0{\cdot}6) = 1{\cdot}738.$$ *6*

(b) Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & 0 < x < \frac{1}{2} \\ 1 - x & \frac{1}{2} < x < 1. \\ 0 & x > 1 \end{cases}$$ *4*
