M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

June, 2022

MMT-003 : ALGEBRA

Time : 2 hours *Maximum Marks : 50*

Note : *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *questions from questions no. 1 to 5. The use of calculators is* ***not*** *allowed.*

1. (a) If H is a proper subgroup of a group G of index p, where p is the smallest prime dividing |G|, then prove that H is a normal subgroup of G. *5*

(b) Obtain the units of $\mathbf{Z}/30\,\mathbf{Z}$. *2*

(c) Prove that the free group on {a} is isomorphic to **Z**. *3*

2. (a) Let K be a field with $p^r = q$ elements, where p is a prime and $r \in \mathbf{N}$.

(i) Show that $GL_n(K)$ acts transitively on $K^n \setminus \{0\}$, $n \in \mathbf{N}$.

(ii) Find the cardinality of the orbit of $(1, 0, ..., 0)$.

(iii) Hence, show that

$$O(GL_n(K)) = q^{n-1} O(GL_{n-1}(K)) (q^n - 1).$$

(iv) Apply the principle of induction to prove that

$$O(GL_n(K)) = (q^{\frac{n(n-1)}{2}}, (q^n - 1)(q^n - 2) ... (q - 1).$$ *6*

(b) Solve the following congruences : *4*

$x \equiv 1 (\text{mod } 3)$, $2x \equiv 3 (\text{mod } 5)$, $x \equiv 6 (\text{mod } 7)$

3. (a) Give an example, with justification, of an injective representation of dimension two of the cyclic group of order 3. *3*

(b) Find the splitting field of the polynomial $x^p - p \in \mathbf{Q}[x]$, where p is a prime. Also find its degree over $\mathbf{Q}$. *7*

4. (a) Prove that any finite extension K of a field F is an algebraic extension. Is the converse true ? Justify your answer. *5*

(b) Calculate the Legendre symbol $\left(\frac{85}{101}\right)$. *3*

(c) Check whether or not $(\mathbf{N}, \cdot)$ is a finitely generated semigroup. *2*

5. (a) List all the finite abelian groups (up to isomorphism) of order 540. *4*

(b) How many Sylow p-subgroups can a group of order 40 have, where p = 2, 3, 5 ? Further, can a group of order 40 be simple ? Give reasons for your answers. *4*

(c) Find Aut $(\mathbf{F}_p)$. *2*

6. State, with reasons, which of the following statements are *True* and which are *False* : *10*

(a) A simple extension of a field is a normal extension.

(b) There exists a non-abelian group of order 49.

(c) The usual action of the symmetric group S_n on {1, 2, ..., n} is transitive.

(d) $SL_2(\mathbf{C}) = S \cup (2)$.

(e) $\mathbf{F}_{p^r}$ is a subfield of $\mathbf{F}_{p^s}$ whenever $r \leq s$.