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MMT-007

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

June, 2021

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 50%)

Note : *Question no.* ***1*** *is* ***compulsory****. Attempt any* ***four*** *questions out of the remaining questions no. 2 to 7. Use of scientific non-programmable calculator is allowed.*

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1. State whether the following statements are *True* or *False*. Justify your answer with the help of a short proof or a counter-example. No marks will be awarded without justification. *5×2=10*

(a) The initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x}, \ y(0) = 1$$

has a unique solution.

(b) If inverse Laplace transform is denoted by $\mathcal{L}^{-1}$, then

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \frac{1}{2}\,(\sin t + \cos t + e^{-t}).$$

(c) If the heat conduction equation $u_t = u_{xx}$ is approximated by the method

$$\frac{1}{2k}\left(u_m^{n+1} - u_m^{n-1}\right) = \frac{1}{h^2}\left(u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}\right),$$

then the order of the method is $O(k^3 + kh)$.

(d) For the boundary value problem

$$y''(x) = 0, \;\; y(0) = y(1), \;\; y'(0) = y'(1),$$

Green's function does not exist.

(e) To solve the boundary value problem

$$(1 + x^2)\, y'' + 4xy' + 2y = 2$$

with $y(0) = 0, \;\; y(1) = \frac{1}{2}$,

using first order difference method, the approximations used are

$$y_i'' = \frac{1}{h^2}\,(y_{i+1} - 2y_i + y_{i-1})$$

and

$$y_i' = \frac{1}{2h}\,(y_{i+1} - y_{i-1}).$$

2. (a) Solve, in series, the differential equation

$xy'' = (1 + x)\, y' + 2y = 0$ about $x = 0$. *6*

(b) Using the generating function for Legendre polynomial P_n, $n = 0, 1, 2, \ldots$, prove that

$$1 + \frac{1}{2} P_1 (\cos \theta) + \frac{1}{3} P_2 (\cos \theta) + \ldots = ln \left[\frac{1 + \sin (\theta/2)}{\sin (\theta/2)} \right].$$ *4*

3. (a) Using convolution theorem, find the Fourier inverse of the functions $\frac{1}{(i\alpha + k)^2}$, $k > 0$. *4*

(b) Find the solution to the initial boundary value problem, subject to given initial and boundary conditions, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$,

$u(x, 0) = 2x$ for $x \in \left[0, \frac{1}{2}\right]$,

$u(0, t) = 0 = u(1, t)$

$u(x, 0) = 2\,(-x)$ for $x \in \left[\frac{1}{2}, 1\right]$

using Schmidt method with $\lambda = 1/6$ and $h = 0 \cdot 2$. *6*

4. (a) Using Fourier transforms, solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \; -\infty < x < \infty, \; t > 0$$

with $u(x, 0) = f(x), \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$. *5*

(b) Reduce the second order initial value problem $y'' = y' + 3$ with $y(0) = 1$ and $y(0) = \sqrt{3}$ to a system of first order initial value problems. Hence find $y(0{\cdot}1)$ and $y'(0{\cdot}1)$ using Taylor series method of second order with $h = 0{\cdot}1$. *5*

5. (a) Find the solution of the boundary value problem

$$\nabla^2 u = x^2 + y^2, \; 0 \le x \le 1, \; 0 \le y \le 1$$

subject to the boundary conditions

$u = \frac{1}{12}(x^2 + y^4)$ on the lines $x = 1$, $y = 0$, $y = 1$ and $12u + \frac{\partial u}{\partial x} = x^4 + y^4 + \frac{x^3}{3}$ on $x = 0$ using the five-point formula. Assume $h = \frac{1}{2}$ along both axes. Use central difference approximation in the boundary conditions. *6*

(b) Find the Fourier cosine transform of

$$f(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ (1 - x), & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}.$$ *4*

6. (a) Solve the boundary value problem

$$y'' = xy,$$

$$y(0) + y'(0) = 1, \; y(1) = 1.$$

Take $h = \frac{1}{3}$ and use second order method. *6*

(b) Obtain the general solution of the differential equation

$$(x + 3)^2 y'' - 4(x + 3) y' + 6y = ln(x + 3).$$ *4*

7. (a) Using the generating function for Legendre polynomial $P_n(x)$, show that

$$\frac{P_{n+1}(x) - P_{n-1}(x)}{2n + 1} = C + \int P_n(x)\, dx,$$

where C is a constant. *6*

(b) Derive the Fourier-Bessel series for $f(x) = x, \; 0 \leq x \leq 1$, in terms of the function $J_1(\lambda_n x)$, where λ_n are the zeros of $J_1(x)$. *4*
