MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)
Term-End Examination
June, 2021

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours | *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *of the remaining questions. Use of calculator is not allowed. Notations are as in the study material.*

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1. (a) Prove that Hahn-Banach extensions are unique in inner product spaces. *3*

(b) Suppose X is a normal linear space and suppose that every absolutely convergent series is convergent. Show that X is a Banach space. Is the converse true ? Justify your answer. *4*

(c) Give an example of a normal operator on a Hilbert space that is not self-adjoint. *3*

2. (a) Let f be a bounded linear functional on a Banach space X. On $G = \{(x, f(x)) : x \in X\}$, define $\| (x, f(x)) \| = \| x \| + | f(x) |$. Prove that $\|\cdot\|$ defines a norm on G. Also show that G is complete w.r.t. this norm. *4*

(b) If A is a bounded linear operator on a Banach space, show that $\sum_{0}^{\infty} \frac{A^n}{n!}$ converges in operator norm. *3*

(c) Let $g \in C[0, 1]$ and define $Af(x) = e^x \cdot \int_0^1 g(y)\, f(y)\, dy$ for $f \in C[0, 1]$. Prove that A is a compact operator on C[0, 1]. *3*

3. (a) State the Uniform Boundedness Principle. Use it to prove the following : If X, Y are Banach spaces and if $A : X \to Y$ is a linear map such that $f \circ A$ is a bounded linear functional on X for every bounded linear functional f on Y, then A is continuous. *4*

(b) Find an orthonormal basis in l^2, $\{u_1, u_2, ...\}$, such that $u_1 = \frac{e_1 + e_2}{\sqrt{2}}$. *3*

(c) Determine a bounded linear functional f on l^4 such that $f(e_{2n}) = 0$ for all n and $\| f \| = 1$. *3*

4. (a) If A is a normal operator on a Hilbert space, prove that there are self-adjoint operators B, C such that A = B + iC and $\| A \|^2 = \| B \|^2 + \| C \|^2$. *4*

(b) Let φ be a bounded linear functional on C [–1, 1] such that φ(hn) = 0, $hn(x) = x^n$, $n \geq 0$. Find φ(h), where $h(x) = |x|$. *3*

(c) Prove that there is no linear isometry from $(\mathbb{R}^2, \|\cdot\|_1)$ onto $(\mathbb{R}^2, \|\cdot\|_2)$, where

$\| x \| = |x_1| + |x_2|$ and

$\| x \| = (|x_1|^2 + |x_2|^2)^{1/2}$, $x = (x_1, x_2) \in \mathbb{R}^2$. *3*

5. (a) State the open mapping theorem. Show that any bounded linear open map between normed linear spaces is surjective. *4*

(b) Let X be a reflexive normed linear space. Show that X is a Banach space. Also show that X' is reflexive. *4*

(c) Consider the operator $A : C[0, 1] \to C[0, 1]$ given by $Af(x) = xf(x)$, $f \in C[0, 1]$. Show that the eigen spectrum of A is empty. *2*

6. State, with justification, whether the following statements are *True* or *False* : *5×2=10*

(a) If f is a bounded linear functional on l^{∞}, then there is an a = (a_n) in l' such that $f(x) = \sum a_n x_n$.

(b) If λ is an eigenvalue of a bounded linear operator A, then $|\lambda| \leq \| A \|$.

(c) If a Banach space X is linearly isometric to a Hilbert space H, then X is also a Hilbert space.

(d) If $A : l' \to l'$ is compact, then $0 \in \sigma(A)$.

(e) A bounded linear operator U on a Hilbert space H is unitary if $\|Ux\| = \|x\|$ for all $x \in H$.