MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2021

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no.* ***1*** *is* ***compulsory****. Attempt any* ***four*** *questions from questions no.* ***2*** *to* ***6****. Calculators are* ***not*** *allowed.*

---

1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) The sequence $\{f_n\}$ in $(C[0, 1], d_\infty)$ given by $f_n(x) = 1 + x^n$, $x \in [0, 1]$, $n = 1, 2 \dots$ is convergent in $C[0, 1]$ under the sup-metric.

(b) The set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ is compact in $\mathbf{R}^2$.

(c) If the function $f : \mathbf{R}^n \to \mathbf{R}^n$ is locally invertible at $x_0 \in \mathbf{R}^n$, then the Jacobian of f is non-zero at x_0.

(d) Let $\mathcal{A}$ be the class of open set $\bar{S}$ in a metric space X. Then there does not exist a σ-algebra containing $\mathcal{A}$.

(e) For any measurable function f,

$$\int |f| \, dm < \left| \int f \, dm \right| .$$

2. (a) Define path connectedness in a metric space. Prove that any continuous image of a path connected set is path connected. *3*

(b) State implicit function theorem. Verify the theorem for the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = y^2 - yx^2 - 2x^5$ in the neighbourhood of the point (1, –1). *4*

(c) Let $f : (X, d_1) \to (Y, d_2)$ be such that $d_2(f(x), f(y) \leq 2d_1 (x, y)$ for $x, y \in X$. Show that f is uniformly continuous. *3*

3. (a) For any measurable set E and for any $\varepsilon > 0$, prove that there exists an open set O containing E such that $m (O \setminus E) < \varepsilon$. *3*

(b) State and prove the dominated convergence theorem. *4*

(c) For a function f in $L^1(\mathbf{R})$, define the Fourier transform $\hat{f}$. Prove that the Fourier transform $\hat{f}$ is a continuous function on $\mathbf{R}$. *3*

4. (a) Prove that if a Cauchy sequence in a metric space has a convergent subsequence, then it is convergent. *4*

(b) Let f be a C^1 function defined on an open set $E \subset \mathbf{R}^n$ to $\mathbf{R}^n$. Suppose $Jf(x) \neq 0$ for all $x \in E$. Prove that f(B) is an open set in $\mathbf{R}^n$ for any open set $B \subset E'$. *4*

(c) Is the set of irrational numbers measurable ? Justify your answer. *2*

5. (a) Does the sequence $\{f_n\}$ where

$f_n = \chi_{[n,\ n+1]}$, n=1, 2... satisfy the conditions of monotone convergence theorem ? Does the conclusion of the theorem hold good for this sequence ? Justify. *4*

(b) Define Stable, Casual systems. Give examples of stable, unstable, casual and non-casual systems. *4*

(c) Let (X, d) be a metric space and A be a subset of X such that $bdy(A) = \phi$. Show that A is both open and closed. *2*

6. (a) Show that every sequence in a compact metric space has a convergent subsequence. *3*

(b) Obtain the second Taylor's series expansion for the function f given by

$$f(x, y) = x^2y + xe^y \text{ at } (1, 0).$$ *4*

(c) Find the outer measure of the set

$E = \{x \in \mathbf{R} : \sin x = 0\} \cup [\frac{1}{2}, 1].$ *3*
