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MMT-008

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

June, 2021

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Note : *(i) Question No. 8 is **compulsory**. Attempt any **six** questions from question nos. **1** to **7**.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

1. (a) Let (X, Y) have the joint p.d.f. given by : 9

$$f(x, y) = \begin{cases} 1, & \text{if } |y| < x; 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal p.d.f.'s of X and Y.

(ii) Test the independence of X and Y.

P. T. O.

(iii) Find the conditional distribution of X given $Y = y$.

(iv) Compute $E(X \mid Y = y)$ and $E(Y \mid X = x)$.

(b) Let $X \sim N_3(\mu, \Sigma)$, where $\mu = [5, 3, 4]'$ and 6

$$\Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0.5 \\ 1 & 0.5 & 1 \end{pmatrix}.$$

Find the distribution of :

$$\begin{pmatrix} 2X_1 + X_2 - X_3 \\ X_1 + X_2 + X_3 \end{pmatrix}$$

2. (a) Determine the principal components Y_1, Y_2 and Y_3 for the covariance matrix :

$$\Sigma = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Also calculate the proportion of total population variance for the first principal component. 9

(b) Consider a Markov chain with transition probability matrix : 6

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{pmatrix} \end{array}$$

(i) Whether the chain is irreducible ? If irreducible classify the states of a Markov chain i. e., recurrent, transient, periodic and mean recurrence time.

(ii) Find the limiting probability vector.

3. (a) At a certain filling station, customers arrive in a Poisson process with an average time of 12 per hour. The time interval between service follows exponential distribution and as such the mean time taken to service to a unit is 2 minutes. Evaluate : 8

(i) Probability that there is no customer at the counter.

(ii) Probability that there are more than two customers at the counter.

(iii) Average no. of customers in a queue waiting for service.

(iv) Expected waiting time of a customer in the system.

(v) Probability that a customer wait for 0.11 minutes in a queue.

(b) Box A contains 4 red, 2 white and 6 black balls and box B contains 3 red and 5 white balls. A fair die is tossed. If 1 or 6 appears, a ball is chosen from box A, otherwise a ball is chose from B. If a red ball is chosen, what is the chance that a 6 appeared on a die ? 7

4. (a) Let $\{X_n\}$ be a branching process where the probability distribution of number of offsprings be geometric. Then find the probability generating function of 2nd generation in G. W. Branching processes with $X_0 = 1$ and its expectation i.e., E (size of second generation). 7

(b) A service station has 5 mechanics each of whom can service a scooter in 2 hours on the average. The scooters are registered at a single counter and then sent for servicing to different mechanics. Scooters arrive at a service station at an average rate of

2 scooters per hour. Assuming that the scooter arrivals are Poisson and service times are exponentially distributed, determine : 8

(i) Identify the model.

(ii) The probability that the system shall be idle.

(iii) The probability that there shall be 3 scooters in the service centre.

(iv) The expected no. of scooters waiting in a queue.

(v) The expected no. of scooters in the service centre.

(vi) The average waiting time in a queue.

5. (a) A random sample of 12 factories was conducted for the pairs of observations on sales (x_1) and demands (x_2) and the following information was obtained : 8

$\Sigma X = 96$, $\Sigma Y = 72$, $\Sigma X^2 = 780$, $\Sigma Y^2 = 480$, $\Sigma XY = 588$

The expected mean vector and variance covariance matrix for the factories in the population are :

$$\mu = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

and $$\Sigma = \begin{bmatrix} 13 & 9 \\ 9 & 7 \end{bmatrix}.$$

Test whether the sample confirms its truthness of mean vector at 5% level of significance, if :

(i) Σ is known,

(ii) Σ is unknown.

[You may use : $\chi^2_{2,0.05} = 10.60$, $\chi^2_{3,0.05} = 12.83$, $\chi^2_{4,0.05} = 14.89$, $F_{2,10,0.05} = 4.10$]

(b) Let X and Y have bivariate normal distribution with parameters : 5

$\mu_X = 5$, $\mu_Y = 10$, $\sigma^2_X = 1$, $\sigma^2_Y = 25$ and corr (X, Y) = ρ.

If $\rho > 0$, find ρ when $P(4 < Y < 16 \mid X = 5) = 0.954$.

Use $P(-2 < Z < 2) = 0.954$.

(c) Using the value of ρ obtained in (b), calculate $V(Y \mid X = 5)$. 2

6. (a) Let the random vector $X' = (X_1, X_2, X_3)$ has mean vector [–2, 3, 4] and variance covariance matrix = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix}$. Fit the equation $Y = b_0 + b_1 X + b_2 X_2$. Also obtain the multiple correlation coefficient between X_3 and $[X_1, X_2]$. 6

(b) Let $\{N_n, n = 0, 1, 2, ...\}$ be a renewal process with sequence of renewal periods $\{X_i\}$. Each X_i follows the Bernoulli distribution with $P(X_i = 0] = 0.3$ and $P(X_i = 1) = 0.7 = p$. Find the distribution of $\{N_n, n = 0, 1, 2, ...\}$. 4

(c) Define ultimate extinction in a branching process. Let $p_k = bc^{k-1}$, $k = 1, 2, ...;$ $0 < b < c < b + c < 1$ and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Then discuss the probability of extinction in different cases for $E(X_1) \geq 1$ or $E(X_1) < 1$. 5

7. (a) If the random vector Z be $N_4(\mu, \Sigma)$, where : 8

$$\mu = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 3 & 3 & 0 & 9 \\ 3 & 2 & -1 & 1 \\ 0 & -1 & 6 & -3 \\ 9 & 1 & -3 & 7 \end{bmatrix}.$$

Find r_{34}, $r_{34.21}$.

(b) Suppose life times $X_1, X_2, ...$ are i.i.d. uniformly distributed on (0, 3) and $C_1 = 2$ and $C_2 = 8$. Find : 7

(i) μ^T

(ii) T which minimizes C (T) and which is the better policy in the long-run in terms of cost.

8. State whether the following statements are true or false. Justify your answer with a short proof or a counter example : 10

(i) If P is a transition matrix of a Markov chain, then all the columns of $\lim_{n \to \infty} p^n$ are identical.

(ii) In a variance-covariance matrix all elements are always positive.

(iii) If X_1, X_2, X_3, X_4 are i.i.d. from $N_2(\mu, \Sigma)$, then $\frac{X_1 + X_2 + X_3 + X_4}{4}$ follows $N_2\left(\mu, \frac{1}{4}\Sigma\right)$.

(iv) The partial correlation coefficients and multiple correlation coefficients lie between –1 and 1.

(v) For a renewal function $M_t, \lim_{t \to 0} \frac{M_t}{t} = \frac{1}{\mu}$.