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MMT-003

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

June, 2021

MMT-003 : ALGEBRA

Time : 2 Hours *Maximum Marks : 50*

Note : *Question **No. 1** is **compulsory**. Attempt any **four** of the remaining questions. Calculators are **not** allowed.*

1. State, with reasons, which of the following statements are true and which are false : 10

   (i) If D is an integral domain, and F and L are fields, s. t. F $\subseteq$ D $\subseteq$ L, then D is a field.

   (ii) Any subgroup of the multiplicative group of non-zero elements of $\mathbf{F}_{13^4}$ must be cyclic.

   (iii) The number of distinct abelian groups of order $p_1^{n_1} p_2^{n_2}$, where p_i are primes and $n_i \in \mathbf{N}$, is $n_1 n_2$.

   (iv) S_6 has 6 elements of order 6.

   (v) 37 is a square modulo 73.

2. (a) Let G = S_4, the symmetric group on 4 symbols. Let G act on G by conjugation, i. e., if g, $a \in G$, then $g * a = gag^{-1}$. What is the orbit of the cycle (1 2) and what is the stabiliser of (1 2) ? 4

   (b) Check whether or not a finite monoid is a group. 2

   (c) Write $\begin{bmatrix} 3 & 3 \\ -3 & 2 \end{bmatrix}$ as a product of an orthogonal matrix and an upper triangular matrix. Clearly show each step used in the process. 4

3. (a) Show that a group of order 108 cannot be simple. 7

(b) Let $\mathbf{F}_{p^m}$ be a subfield of $\mathbf{F}_{p^n}$, where p is a prime. Show that m divides n. 3

4. (a) Let a be a generator of the cyclic group $\mathbf{F}_p^*$, where p is a prime. Show that $\left(\frac{a}{p}\right) = -1$, ($p$ is odd). 5

(b) If G = $(\{a, b\}, \{g_0\}, \{g_0 \to b, g_0 \to ab\}, g_0)$, find the language generated by G. 2

(c) Show that every semigroup is the homomorphic image of a free semigroup. 3

5. (a) Is $Q(\sqrt[7]{5})$ a Galois extension of Q ? If so, what is the Galois group of $Q(\sqrt[7]{5})$ over Q ? If not, what is the smallest extension of $Q(\sqrt[7]{5})$ which is Galois over Q ? What is the degree of this extension over Q ? Justify your answers. 7

(b) Write the class equation of a group G of order n. Is there a group of order 10 with class equation 10 = 1 + 2 + 2 + 5 ? If yes, exhibit the group and its conjugacy classes. If no, give a class equation of a group of order 12. 3

6. (a) Solve the following simultaneous system of congruences : 4

$$x \equiv 2 \,(\text{mod } 5),\ x \equiv 1 \,(\text{mod } 7),\ x \equiv 3 \,(\text{mod } 11).$$

(b) Give an example, with justification, of an abelian group of rank 8 and with torsion group being non-cyclic of order 8. 2

(c) Find a free group F and $N \Delta F$, such that $D_{12} \simeq \frac{F}{N}$. 4