No. of Printed Pages : **4** **MMT-006**

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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

June, 2016

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *(i) Question no. 6 is **compulsory**.*

*(ii) Attempt **any four** of the remaining questions.*

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1. (a) Define equivalence of norms. Prove that all norms defined on a finite-dimensional linear space are equivalent. **3**

(b) Consider the space C $[-1, 1]$ of real-valued continuous functions on $[-1, 1]$, with the inner product $<, >$, defined by **3**

$$<x, y> = \int_1^1 x(t)\, y(t)\, dt, (x, y) \in [-1, 1].$$

If M is the subspace of even functions in C $[-1, 1]$, show that every odd function in C $[-1, 1]$ is in $M^{\perp}$. Further, find the norm of the identity function in C $[-1, 1]$.

(c) Define the operator $A : C^3 \to C^3$ by $A(z_1, z_2, z_3) = (iz_1, e^{2i}z_2, z_3)$. Check whether A is : 4

(i) self-adjoint;

(ii) unitary

2. (a) Consider $\mathbf{R}^2$ with $||.||_2$. Let $M = \{(x_1, x_2) \in \mathbf{R}^2 | x_1 = x_2\}$ and $x = (-1, 1)$. Find $d(x, M)$. 2

(b) Let $F : X \to Y$ be a linear map between two normed spaces. Prove that F is continuous at O if and only if F is uniformly continuous on X. 4

(c) Define the spectrum and the approximate eigen spectrum of an operator A in BL (H), where H is a Hilbert Space. 2

(d) Let X = C [0, 1] with the sub norm. Show that there exists T in BL (X) whose spectrum is [0, 1]. 2

3. (a) For a normed space X, prove that the dual of X is separable implies that X is separable. Is the converse true ? Give reasons for your answer. 4

(b) Prove that l^p is not a Hilbert Space, where $p>2$. 3

(c) Give an example, with justification of an orthonormal basis of $L^2([-\pi, \pi])$. 3

4. (a) Suppose A is a non-zero compact self-adjoint operator on a Hilbert space H over K. Prove that there exists a finite set $\{r_1, r_2, \ldots\ldots r_n\}$ of non-zero real numbers with $|r_1| \geq |r_2| \geq |r_3| \geq \ldots\ldots \geq |r_n|$ and an orthonormal set $\{w_1, w_2, \ldots\ldots w_n\}$ in H such 5

that $A(x) = \sum_{i=1}^{n} r_i < x, w_i > w_i, x \in H$.

(b) Let H be a Hilbert space and $f \in H'$. Show that there exists one and only one $y \in H$ such that $f(x) = <x, y> \ \forall x \in H$. 5

5. (a) Let X be a normed space over K. For $x \in X$, define $f_x : x' \to K$ by $f_x(x') = x'(x)$, $x' \in X'$. Show that $f_x \in X''$ and $\|f_x\| = \|x\| \ \forall x \in X$. 4

(b) Let X, Y be Banach spaces and $F : X \to Y$ be a linear map which is continuous and open. Will F always be closed ? Will F always be surjective ? Give reasons for your answers. 3

(c) Check whether the identity map on an infinite-dimensional normed space is compact. 3

6. Which of the following statements are true, and which are false ? Give reasons for your answers. **10**

(a) $\mathbf{R}^n$, $n \geq 2$, is an infinite-dimensional normed space.

(b) $(C_{00}, \|\cdot\|_\infty)$ is a Banach space.

(c) Every normal operator on a Hilbert space H is a unitary operator on H.

(d) The space l^1 is reflexive.

(e) If a linear subspace Y of a normed space X has a non-empty interior, then $Y = X$.