M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00996

Term-End Examination

June, 2016

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *questions out of questions no. 2 to 7. Calculators are* ***not*** *allowed.*

1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) Arbitrary union of compact sets is compact.

(b) If $f : \mathbf{R} \to \mathbf{R}$ is the function given by $f(x) = x^2 + 4x + e^x$ and A is the interval (0, 5), then f(A) is connected.

(c) A subset of a set X which is open with respect to one metric on X will be open with respect to every other metric on X.

(d) The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x, y) = (x^2, y^2 - 1)$ is invertible near (0, 0).

(e) The domain of the Lebesgue outer measure is the power set of the rationals.

2. (a) Find the directional derivation of the function $f : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$f(x, y, z, w) = (x^2y, xyz, x^2 + y^2, zw^2)$

at the point (1, 2, – 1, – 2) in the direction $v = (1, 0, -2, 2)$. *3*

(b) Suppose that $f : \mathbf{R} \to \mathbf{R}^2$ is given by $f(t) = (t, t^2)$ and $g : \mathbf{R}^2 \to \mathbf{R}^3$ is given by $g(x_1, x_2) = (x_1^2, x_1x_2, x_2^2 - x_1^2)$. Compute the derivative of g o f. *3*

(c) Find the outer measure of the following sets. Also state the results used in computing the outer measures. *4*

(i) $A : \{x^2 - 1 : 1 \le x \le 2\} \cup [4, 5]$

(ii) $S = \left\{\frac{1}{2^n} : n \in \mathbf{N}\right\}$

3. (a) Show that the projection map $p : \mathbf{R}^2 \to \mathbf{R}$ given by $p(x, y) = y$ is uniformly continuous. *3*

(b) Suppose that E is an open subset of $\mathbf{R}^n$. When is a map $f : E \to \mathbf{R}^m$ said to be differentiable on E ?

If $f : E \subset \mathbf{R}^n \to \mathbf{R}^m$ is a function such that the partial derivatives of f exist and are continuous on E, then show that f is differentiable on E. *4*

(c) Is the continuous image of a Cauchy sequence a Cauchy sequence ? Justify. *3*

4. (a) Obtain the Taylor's series expansion up to 2^{nd} derivative for the function given by

$$f(x_1, x_2) = \sin(x_1 + x_2) \text{ at } \left(0, \frac{\pi}{2}\right).$$ *4*

(b) Define the Lebesgue integral of a non-negative measurable function over a measurable set. If E is a measurable set and f is a measurable function such that

$a \leq f(x) \leq b \;\; \forall\, x \in E$, show that

$$a.m(E) \leq \int_E f(x)\, dm \leq m(E).b.$$ *4*

(c) Find $B\left[2, \frac{1}{2}\right]$ in $(\mathbf{R}, d)$, where d is the metric given by $d(x, y) = \min\{1, |x - y|\}$. *2*

5. (a) (i) Give an example of a family $\mathcal{F}$ of subsets of a set X which has finite intersection property. Justify your choice of example.

(ii) If every collection of closed subsets of a metric space X with finite intersection property has non-empty intersection, then show that X is compact. *5*

(b) State the inverse function theorem. Prove that the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ given by $f(x, y, z, w) = (x, x - y, y^2z, zw)$ is locally invertible at the point (1, 1, 1, 0). *3*

(c) Check whether the collection

$$U = \left\{B\left(n, \frac{1}{2}\right); n \in \mathbf{Z}\right\}$$

forms an open cover of **R**. *2*

6. (a) Suppose f is integrable on $[-\pi, \pi]$ and is Lipschitz continuous at $\theta \in [-\pi, \pi]$. Show that the Fourier series of f converges to f at θ. *3*

(b) Let (X, d) be a metric space and A be a subset of X. Show that $\text{bdy}(A) = \phi$ if and only if A is both open and closed. *3*

(c) Find the Fourier sine series expansion of the function

$$f(t) = \begin{cases} \frac{1}{2}, & 0 < t < \frac{1}{2} \\ 0, & \frac{1}{2} \le t < 1 \end{cases}$$ *4*

7. (a) Verify Fatou's lemma for the sequence $\{f_n\}$ given by

$$f_n(x) = 2n, \quad \text{for } x \in \left(\frac{1}{2n}, \frac{1}{n}\right)$$

$$= 0, \quad \text{for } x \in \left(0, \frac{1}{2n}\right) \cup \left(\frac{1}{n}, 1\right).$$ *4*

(b) Prove that the set of irrationals is not a connected set, considered as a subset of **R** with the usual metric. *3*

(c) Define a stable system. Check whether the system $\mathcal{R} : S \to S$ given by

$$g(t) = (\mathcal{R}f)(t) = \int_{-\infty}^{t} f(\tau)\, e^{-(t-\tau)} d\tau$$

is stable or not, where S denotes the set of signals. *3*