No. of Printed Pages : 7 **MMT-008**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

June, 2013

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question number 8 is **compulsory**. Answer **any six** questions from question number **1** to 7. Use of calculator is **not** allowed.*

1. (a) Consider two random variables X and Y whose joint probability mass function is given in the following table. **10**

x \ y	-5	1	4
1	0.15	0.15	0.2
2	0.05	0.05	0.1
4	0.1	0.1	0.1

(i) Find E(X), E(Y), V(X) and V(Y)

(ii) Are X and Y independent ? Give reasons.

(iii) Find $E(X/Y=4)$ and $V(X/Y=4)$

(iv) Find Cov (X, Y)

(b) Let $X = (X_1 \ X_2 \ X_3)' \sim N_3(\mu, \Sigma)$ where 5

$\mu = (-1 \ 1 \ -1)'$ and $\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. Let

$Y = \begin{pmatrix} 2X_1 + X_2 + X_3 \\ X_1 + 2X_2 + X_3 \end{pmatrix}$. Find $l_{2 \times 1}$ such that

$l'y \sim N(0, 1)$.

2. (a) Consider the mean vector $\mu_X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and 8

$\mu_Y = 3$ and the covariance matrices of

$(X_1 \ X_2)'$ and Y are $\Sigma_{xx} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\sigma_{yy} = 14$

and $\sigma_{xy} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(i) Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean squared error.

(b) Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition probability **7**

$$\text{matrix} \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the communicating classes.

(ii) Find all stationary distributions.

3. (a) Three friends Ashish, Basant and Chetan occupy rooms numbered 1, 2, 3 in a hostel. Another friend Falguni stays with them alternating among the three rooms. He never stays in the same room on two consecutive days. Everyday he chooses one of the two available rooms at random. Let X_n be the room number he stays in on day n. **10**

(i) Show that X_n is a Markov chain.

(ii) Find the transition probability matrix.

(iii) If Falguni starts (on day 0) in room number 3, find the probability that the next time he stays in the same room is on day n.

(iv) Find the mean recurrence time for room 3.

(b) Suppose the interoccurance times $\{x_n : n \geq 1\}$ are uniformly distributed on $[0, 1]$. 5

(i) Find $\widetilde{M}_t$, the Laplace transform of the renewal function, M_t.

(ii) Find $\lim_{t \to \infty} M_{t/t}$.

4. (a) Consider a branching process with offspring distribution given by 7

$$p_j = \begin{cases} 1/3 & j = 0 \\ 2/3 & j = 2 \end{cases}.$$

Find the probability of extinction.

(b) Determine the definiteness of the quadratic form $2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1 x_3$. 5

(c) Define conjoint analysis. Give two applications of conjoint analysis. 3

5. (a) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval [2, 5] years. Further, planned replacements take place every 3 years. Compute 6

(i) long - term rate of replacements.

(ii) long - term rate of failures.

(iii) long - term rate of planned replacements.

(b) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2 where $X_1 \sim N_2(\mu^{(1)}, \Sigma)$ and $X_2 \sim N_2(\mu^{(2)}, \Sigma)$ 9

$\mu^{(1)} = (1\ 2)'$, $\mu^{(2)} = (-1\ 0)'$

and $\Sigma = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}'$.

Considering equal cost and equal prior probabilities, classify the observation $(-1\ 1)'$ in one of the two populations.

6. (a) Two samples of sizes 40 and 60 respectively were drawn from two different lots of a certain manufactured component. Two characteristics X_1 and X_2 were measured for the sampled items. The summary statistics of the measurements for lots 1 and 2 is given below. 8

$$\overline{X}_1 = \begin{pmatrix} 6 \\ 3 \end{pmatrix};\ \overline{X}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix};\ S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix};$$

$S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Assume normality of X_1 and X_2 and that $\Sigma_1 = \Sigma_2$. Test at 1% level of significance whether $\mu_1 = \mu_2$ or not. You may like to use the following values.

$$\begin{pmatrix} F_{2,97}(0.01) \approx 4.86 \\ F_{3,97}(0.01) \approx 4.05 \end{pmatrix}.$$

(b) Consider three random variables X_1, X_2, X_3 having the following covariance matrix. **7**

$$\begin{pmatrix} 1 & 0.12 & 0.08 \\ 0.12 & 1 & 0.06 \\ 0.08 & 0.06 & 1 \end{pmatrix}$$ where no. of variables

$\lambda(p) = 3$ and no. of factors '$(m) = 1$. Write the factor model.

7. (a) Customers arrive at a fast food counter in a Poisson manner at an average of 40 customers per hour. The service time per customer is exponential with mean 1 minute. What is the **10**

   (i) probability that an arriving customer can go directly to the counter to place the order ?

   (ii) probability that there are at least 3 customers in the queue ?

   (iii) average queue length ?

   (iv) expected waiting time for a customer in the system ?

   (v) probability that a customer has to wait at least three minutes in the queue ?

(b) Let N_t be a Poisson process with parameter $\lambda > 0$. Fix $s > 0$ and let the renewal function $M_t = N_{t+s} - N_s$. Show that the conditional distribution of M_t given $N_s = 10$ is Poisson and identify its parameter. **5**

8. State whether the following statements are **true** or **false**. Justify your answer with valid reasons : 10

(a) For independent events B and C with $P(B \cap C) > 0$, we have $P(A/B \cap C) = P(A/B).P(A/C)$.

(b) If X and Y are two random variables with $V(X) = V(Y) = 2$, then $-2 < \text{cov}(X, Y) < 2$.

(c) The row sums in the infinitesimal generator of a birth and death process are zero.

(d) A real symmetric matrix $(aij)_{n \times n}$ with $a_{11} = -1$ cannot be positive definite.

(e) The maximum likelihood estimator of $\mu' \Sigma^{-1} \mu$ is $\overline{X}' U^{-1} \overline{X}$ where $\overline{X}$ and U are the maximum likelihood estimators of μ and Σ respectively.