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MMT-004

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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2013

MMT-004 : REAL ANALYSIS

Time : 2 hours | *Maximum Marks : 50*

Weightage : 70%

Note : *Question No.* 1 *is* ***compulsory***. *Do* ***any four*** *questions out of questions no.* 2 *to* 7.

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1. State, whether the following statements are ***True*** or ***False***. Give reasons for your answers : **5x2=10**

   (a) The set $F = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ in **R** with usual metric, has a limit point in F.

   (b) The characteristic function of the set of all rational numbers in [0, 1] is Lebesgue measurable.

   (c) Every Lebesgue integrable function is Riemann integrable.

   (d) An arbitrary intersection of open sets in a metric space is open.

   (e) In a metric space, every bounded sequence is a Canchy Sequence.

2. (a) Let X = C[0, 1], the set of all real valued functions on [0, 1] which are continuous. For f, g $\in$ C [0, 1], define d : X $\times$ X$\to$**R** by : 3

$$d(f, g) = \int_0^1 |f(t) - g(t)|\, dt,$$ where the integral is Riemann integral. Show that d is a metric on X.

(b) State Monotone Convergence Theorem. Show that the sequence $\{f_n\}$, defined by $f_n = \chi_{[n,\, n+1)}$, n $\in$ N, does not satisfies the conditions of the theorem. Also show that in this case the conclusion of Monotone convergence theorem does not hold. 3

(c) Let E be an open subset of $\mathbf{R}^4$, f map E into $\mathbf{R}^3$ and $x \in$ E. Define differentiability of f at x. Compute f'(a) at a = (1, 2, -1, -2), where $f : \mathbf{R}^4 \to \mathbf{R}^3$ is given by : 4

$f(x, y, z, w) = (x^2yw, y^2z, z^2x)$.

3. (a) Show that every compact metric space is complete. Is it totally bounded ? Justify your answer. 5

(b) Obtain the second order Taylor's series expansion for the function : 5

$f(x_1, x_2) = x^3, x_2 - 4x, e^{x_2}$ at (1, 0).

4. (a) Show that continuous image of a path connected set in path connected. 2

(b) Which of the following sets are connected ? Justify your answer. 3

(i) $A = \{(x, y) \in \mathbf{R}^2 : x \geqslant 1 \text{ and } y = 1\}$

(ii) $B = \{ (x, y) \in \mathbf{R}^2 : x^2 + y^2 = 3\}$

(iii) $C = (0, 1) \cup (2, 3) \subseteq \mathbf{R}$

(c) Let A be a countable set. Show that : 3+2=5

(i) $m^*(A) = 0$.

(ii) A is measurable.

5. (a) If a set E has finite measure. Then show that $L^2(E) \subset L^1(E)$. 3

(b) Let (X, d) be a metric space $a \in X$ be a fixed point of X . Show that the function $f_a : X \to \mathbf{R}$ given by : 3

$f_a(x) = d(a, x)$ is uniformly continuous on X.

(c) Let X be a connected metric space. Then, prove that, any continuous function f from X to the discrete metric space $\{1, -1\}$ is a constant function. 4

6. (a) Find the critical points of the function $f : \mathbf{R}^3 \to \mathbf{R}$, defined by : 5

$f(x, y, z) = x^2y^2 + z^2 + 2x - 4y + z$. Also check whether they are extreme points.

(b) Let $f \in L'(\mathbf{R})$ and $\alpha, \lambda \in \mathbf{R}$. Show that 5

(i) if $g(x) = f(x)e^{i\alpha x}$ then :

$$\hat{g}(w) = \hat{f}(w - \alpha)$$

(ii) if $g(x) = f\left(\frac{x}{\lambda}\right)$, $\lambda > 0$ then $\hat{g}(w) = \lambda \hat{f}(\lambda w)$

where $\hat{f}$ and $\hat{g}$ denote the Fourier transforms of f and g respectively.

7. (a) Compute the Fourier series of the function f given by : 3

$$f(t) = \begin{cases} -2, & -\pi < t < 0 \\ 2, & 0 < t < \pi \end{cases}$$

(b) Show that the system $R : f \to g$ given by : 3

$$g(t) = (Rf)(t) = \int_{-\infty}^{t} f(c)\, dc, \; f \in L'(R)$$

is a time invariant system.

(c) Define the following in the context of signals and systems. Give one example for each : 4

(i) Invertible system

(ii) Causal system.