M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination 00453

June, 2012

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

Note : *Question number 8 is **compulsory**. Answer any **six** questions from question number **1 to 7**. Use of calculator is **not** allowed.*

---

1. (a) Describe birth and death process. If $\lambda_k = \lambda$ and $\mu_k = k_u$, $k \geq 0$, λ, $\mu > 0$ then show that the stationary distribution of the process always exists. Obtain steady state distribution of the process. 8

   (b) Using following transition matrix for a Markov chain find : 7

   (i) Whether the chain is irreducible ? Why ?

   (ii) Probabilities of ultimate return to the states.

   (iii) Mean recurrence times of the states.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

2. (a) Consider a branching process $\{X_n\}$. Given $X_0 = 1$ and probability distribution of number of offsprings to any individual is geometric. Find the probability generating function (p.g.f) of $\{X_n\}$. 5

(b) Find P^n and its limiting value if any for the P-matrix of a Markov chain given below. 7

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

(c) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. The number of people in each family is independent and takes on values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and variance of the number of individuals migrating to this area during a fixed five week period. 3

3. (a) In a city 20% of the population were infected from TB. A diagnostic test reports positive in 95% cases when performed on TB infected person and reports positive in 15% cases when performed on non-infected person. An individual was choosen at random from the city and the test was performed. What is the probability that the result of the test was positive ? If it is positive report then what is the probability that the choosen individual was infected from TB ? 8

(b) Customers arrive in a bank according to poission law at a rate 2 per five minutes. Service time in the bank follows exponential distribution with mean 2 minutes. Find : 7

(i) probability that bank is empty of customers.

(ii) average number of customers in the bank when a customer arrives.

(iii) expected time spent by a customer in the bank.

4. (a) The joint p.d.f of two random variables X and Y is given by : 8

$$f(x,y) = \frac{9\,(x + y + 1)}{2(1 + x)^4\,(1 + y)^4}\,; \qquad 0 \leq x < \infty$$

$$0 < y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for $X = x$.

(b) In a renewal process, renewal period X_n is iid Brenoulli (p). Show that the distribution of N_t will be negative binomial. What will be the renewal function for this process ? 7

5. (a) Let the vector y be distributed as $N_3\,(\mu, \Sigma)$, 8

$$\text{where } \mu = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 3 & -3 & -1 \\ -3 & 6 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(b) Find a, b, c for which matrix A will be orthogonal : 7

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & a \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & b \\ 0 & \frac{1}{\sqrt{2}} & c \end{pmatrix}$$

8. State whether following statements are *true* or *false*. Justify your answer. **2x5=10**

(a) If two events A and B are non-null and mutually exclusive, then both the events are independent.

(b) Sum of elements of a 2×2 transition matrix of a Markov Chain is 4.

(c) The quadratic form $Q = 2x_1^2 - 3x_2^2 - 6x_1 x_2$ is negative definite.

(d) Principal components depend on the scales used to measure the variables.

(e) If $X \sim Np(\mu, \Sigma)$ then the linear combinations of the components of X are normally distributed.

Find :

(i) marginal distribution of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$

(ii) conditional distribution of y_1 given y_2, y_3 and of $y_1\ y_2$ given y_3.

(iii) r_{12} and the partial correlation coefficient $r_{12..3}$.

(b) From the samples of sizes 80 and 100 from two populations following summary statistics were obtained. 7

$$X_1 = \begin{pmatrix} 8 \\ 4 \end{pmatrix} X_2 = \begin{pmatrix} 10 \\ 4 \end{pmatrix} S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

Where X_1 and X_2 are the means and S_1 and S_2 are the standard deviations of two populations. Test equality of population means at 5% level of significance. Assume $\Sigma_1 = \Sigma_2$ given.

You may like to use the following values.

$F_{2,\,177}^{(.05)} = 3.04$ $\quad$ $F_{2,100}^{(0.05)} = 3.1$

$F_{2,\,80}^{(0.05)} = 3.15$

6. (a) $y \sim N_3\,(\mu, \Sigma)$ where : 8

$$\mu = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Obtain

(i) distribution of $c\,y$ where ,

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

(ii) a linear combination $Z = l'\,y$ such that $Z \sim N\,(0, 1)$.

(b) Evaluate Hotelling's T^2 for testing Ho : $\mu' = [8, 10]$ from a sample expressed by the following data matrix : 7

$X = \begin{pmatrix} 2 & 8 & 6 & 8 \\ 12 & 9 & 9 & 10 \end{pmatrix}$ and specify the distribution of T^2.

7. (a) Variance -covariance matrix of three variables X_1, X_2 and X_3 is : 8

$$\Sigma = \begin{pmatrix} 20 & 6 & 5 \\ 6 & 31 & 4 \\ 5 & 4 & 43 \end{pmatrix}$$

The eigen values and corresponding eigen vectors are :

$\lambda_1 = 45.9 \;\; a_1^{-1} \;\; (0.25, 0.34, 0.90)$

$\lambda_2 = 31.1 \;\; a_2^{-1} \;\; (-\,0.28, -\,0.87, 0.41)$

$\lambda_3 = 17.0 \;\; a_3^{-1} \;\; (-\,0.92, 0.36, 0.12)$

(i) Obtain principal components.

(ii) Obtain variances of principal components

(iii) Show that total variation explained by principal components is equal to the total variances of the variables.

(iv) Obtain proportion of variations explained by I^{st} two components.