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M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. SC. (MACS)]

Term-End Examination

December, 2023

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Weightage : 50%

Note : *(i) Question No.* ***8*** *is compulsory. Attempt only* ***six*** *questions from question nos.* ***1*** *to* ***7****.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

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1. (a) Determine the principal components y_1, y_2 and y_3 for the following covariance matrix :

$$\Sigma = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Also calculate the proportion of total population variance for the first principal component. 8

(b) Find the probability of ultimate extinction for the branching process of : 7

$$p_k = (1-a)(1-c)c^{k-1}; \quad k = 1, 2....$$
$$= 9 \qquad ; \quad k = 0$$

for all cases when $a > c$ or $a < c$.

2. (a) Let (X, Y) have the joint p. d. f. given by : 8

$$f(x, y) = e^{-y}; \quad 0 < x < y < \infty .$$

(i) Find the marginal p.d.f's of X and Y.

(ii) Find the conditional distribution X/Y = y.

(iii) Compute E(X/Y = y), E(Y/X = x) and Var (X/Y = y).

(iv) Also find the correlation coefficient between X and Y.

(b) Let X be random vector with : 7

$$E(X) = (1, 2, -1)', \ \Sigma = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Obtain a linear combination $l'x$ which is constant with probability. Find the value of this constant.

3. (a) The transaction probability matrix of a Markov chain $\{X_n; n \geq 1\}$ having the states 1, 2 and 3 is : 9

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

with initial distribution

$$\Pi_0 = (0.7, 0.2, 0.1)$$

Find :

(i) $P(X_2 = 3)$

(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

(iii) Classify the states of a Markov chain.

(iv) Find the limiting probability vector.

(b) A large no. of people were asked to rate their liking of each of the five attributes (taste, money, flavour, snack and energy). The factor loadings of first two factors, obtained through factor analysis are given ahead : 6

(i) Compute the communality of each attribute/variable and percentage of its variance that explained by factors.

(ii) Compute the percentage of total variation explained by each factor separately and both factors jointly.

(iii) Interpret the loading coefficients, variance summarised and communality value obtained above :

Data		
	Factor I	Factor II
Taste	0.02698	0.98545
Money	0.87337	0.00342
Flavour	0.13285	0.97054
Snack	0.81781	0.40357
Energy	0.97337	– 0.1782

4. (a) A shop in the mall has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. 8

Find :

(i) The probability that an arriving customer has to wait for service.

(ii) Average no. of customers in the system.

(iii) Average time spent by a customer in the shop.

(iv) Identify the model.

(b) In investigation of the relationship between three variables X_1, X_2 and X_3, the sample correlation matrix is obtained as given below : 7

$$n = 10, p = 2$$

$$R = \begin{bmatrix} 1 & -0.4 & -0.51 \\ -0.4 & 1 & 0.81 \\ -0.51 & 0.81 & 1 \end{bmatrix}$$

Test the hypothesis whether variables are correlated. You may use the values :

$$\chi^2_{1,0.05} = 2.71$$

$$\chi^2_{2,0.05} = 5.99$$

5. (a) A coin is tossed, p being the probability of head in a toss. Let $\{X_n; n \geq 1\}$ have two states 0 or 1 according as the accumulated no. of head and tails in n tosses are equal or unequal. Show that the states are transient when $p \neq \frac{1}{2}$ and recurrent null when $p = \frac{1}{2}$. 7

(b) Find the steady state solution in L.P.P. for pn; $n \geq 0$ when $\lambda_n = \lambda$ for $n \geq 0$ and $\mu_n = n\mu$; $n > 1$.

Problems arrive at a computing centre in Poisson fashion at a average rate of 5 per day. The rules of computing centre are that any man waiting to get his problem solved must add the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of $\frac{1}{3}$ per day and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line. 8

6. (a) Suppose $n_1 = 10$ and $n_2 = 15$ observations are made on two random variables X_1 and X_2. Given : $\mu^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mu^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 3 & -1 \\ -1 & 7 \end{pmatrix}$, considering equal cost and equal prior probabilities. Check whether the observation (0, 1) belongs to population π_1 or π_2. 8

(b) $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with parameter λ_1, λ_2 respectively, then show that : 5

$$P[N_1(t) = K \mid N_1(t) + N_2(t) = n] = \binom{n}{k} p^k q^{n-k}$$

where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}, q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$.

(c) Find the Laplace transform of a random variable X with density function : 2

$$f(x) = \lambda e^{-\lambda x}; \quad \lambda > 0; x > 0$$

Hence find its mean and variance.

7. (a) Find the mean vector $\overline{X}$ and sample correlation matrix $\hat{R}$ for the data matrix : 8

$$X = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 4 & 3 \\ 3 & 2 & -2 \end{bmatrix}$$

(b) Consider the random vector Z be $N_4(\mu, \Sigma)$, a where $\mu = (2 \; 1 \; 3 \; 4)'$ and : 7

$$\Sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ 1 & -1 & 9 & -1 \\ 1 & -1 & -1 & 16 \end{pmatrix}$$

If Z is partitioned as $Z = (Y_1, Y_2, X_1, X_2)$ then find E(Y/X), Cov (Y/X), r_{12} and $r_{12.34}$.

8. State whether the following statements are true *or* false. Justify your answer with a short proof or a counter-example : 10

   (i) In correlation matrix R all off diagonal elements are always positive.

   (ii) If X_1, X_2, X_3 and X_4 are i.i.d from $N_2(\mu, \Sigma)$, then $(X_1 + 2X_2 + 3X_3 + 4X_4)$ follows $N_2(10\mu, 10\Sigma)$.

   (iii) The partial correlation coefficient and multiple correlation coefficient alway lies between 0 and 1.

   (iv) For a renewal function $M(t)$, $\lim\limits_{t \to \infty} \dfrac{M_t}{t} = \dfrac{1}{\mu}$.

   (v) The general queuing system M/M/K/N represents arrival follow Poisson process, service times follow any general distribution except Poisson or exponential, multiserver queue with finite population.