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MMT–006

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

[M. Sc. (MACS)]

Term-End Examination

December, 2023

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No. **1** is compulsory.*

*(ii) Answer any **four** questions from Question Nos. **2** to **6**.*

(iii) Notations are as in the study material.

1. State whether the following statements True *or* False ? Justify your answers : 5×2=10

(a) $\|.\|$ defined on $\mathbf{R}^n$ as :

$$\|x\| = \sum_{j=1}^{n} |a_j| \text{ for } n = (a_1, a_2, \ldots\ldots, a_n) \in \mathbf{R}^n$$

is a norm.

P. T. O.

(b) C_0 is a Banach space.

(c) If A is the right shift operator on l^2, then the eigen spectrum is non-empty.

(d) If a normed space is reflexive, then so is its dual space.

(e) If a normed linear space X is finite dimesnional, then so is X'.

2. (a) Prove that any *two* norms on a finite dimensional normed space are equivalent. 3

(b) A prove that a Banach space cannot have a denurable basis. 4

(c) Define from A : (C [0, 1], $\|.\|_\infty$ → (C [0, 1], $\|.\|_\infty$) as :

$$A(f)(t) = \int_0^1 sf(s)\,ds$$

show that A is a bounded linear operator and calculate $\|A\|$. 3

3. (a) State closed graph theorem and give an example to show that the theorem fails if the Banach space in replaced by a normed linear space. 5

(b) Prove that a normed linear space which is linearly isometric with a reflexive space is ifself reflexive. 5

4. (a) Use Hahn-Banach extension theorem to prove that if x_0 is a non-zero vector in a normed space X, then there exists an $f \in X'$ such that $f(x_0) = \|x_0\|$ and $\|f_0\| = 1$. 3

(b) Show that a subspace Y of a Hilbert space is dense if and only if $Y^{\perp} = \{0\}$. 4

(c) Let X be a normed linear space and p : X $\rightarrow$ X be a projection. If p is closed, then show that R $\subset$ p and Z (p) are closed.

3

5. (a) Let Y be a closed subspace of a normed linear space X. If both Y and X/Y are Banach spaces, then show that X is complete. 3

(b) Prove that the sequence $\{u_n\}$ in l^2 defined as $u_n = (0, U, ..., 1, 0, 0, ...)$, where 1 occurs at the nth place is an orthonormal basis for l^2. 3

(c) If f is a bounded linear functional on a Hilbert space H, show that there is a unique $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$. 4

6. (a) Give an example of a compact linear map on l^2. 3

(b) Let $\frac{1}{p}+\frac{1}{q}=1$. If $f \in L^p([0,1])$, then show that it defines a bounded linear functional on L^q ([0, 1]). 5

(c) Give an example of a postive operator on ($\mathbf{C}^n$, $\|.\|_2$). 2