No. of Printed Pages : 4 **MMT-004**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

December, 2023

MMT-004 : REAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No.* ***1*** *is compulsory.*

(ii) Attempt any ***four*** *questions from Q. Nos.* ***2*** *to* ***6****.*

(iii) Calculator is not allowed.

(iv) Notations as in the study material.

---

1. State whether the following statements are True *or* False. Give reasons for your answers : 5×2=10

(a) If X is any non-empty set and d_1 is any metric and d_2 discrete metric on X, then any function $f:(X, d_1) \to (X, d_2)$ is continuous.

(b) If (X, d) be a metric space and $x \in \text{X}$, $\text{E} \subset \text{X}$, then $x \in \bar{\text{E}}$ implies that $d(x, \text{E}) = 0$.

(c) $\{0\} \cup \left\{\frac{1}{n}, n = 1, 2, 3.....\right\}$ is compact in **R**.

(d) The function $f(x, y) = xy - 1$ can be solved for x in term of y near 0.

(e) Every L^{-1}-function on [0, 1] is also an L^2-function on [0, 1].

2. (a) Define the diameter of a set. Find the diameter of the following subset of **R** : 2

$$\text{A} = \left\{-\frac{3}{2}, -1, 0, 2, 5, 7\right\}$$

(b) Show that the points (1, 1, 1) and (–1, –1, –1) are the only stationary points of the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by :

$$f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz$$

Check whether the function has maximum or minimum at these points. 4

(c) If A and B are two measurable sets, show that A ∪ B is also measurable. Find the relation $m(\text{A} \cup \text{B}), m(\text{A})$ and $m(\text{B})$. 4

3. (a) (X, d_1) and (Y, d_2) are metric spaces and $f: X \to Y$ is continuous at a point $c \in X$. Prove that given any open set V containing $f(c)$ there exists an open set u containing c in X such that $f(u) \subset V$. 3

(b) Find the directional derivative of the function $f: \mathbf{R}^4 \to \mathbf{R}^4$ defined by :

$$f(x, y, z, w) = \left(x^2 y, xyz, x^2 + y^2, zw^2\right)$$

at the point (1, 2, –1, –2) in the direction $v =$ (1, 0, – 2, 2). 3

(c) Suppose f is a simple measurable function defined on a measurable set E in **R**. Define $\int_E f dm$. If $a \le f(x) \le b$ for all $x \in E$, prove that : 4

$$am(E) \le \int_E f dm \le bm(E)$$

4. (a) State Cantor's Intersection theorem. Prove that the theorem fails for incomplete metric space. 4

(b) Find the derivative of $f: \mathbf{R}^2 \to \mathbf{R}^2$ defined by :

$$f(x, y) = \left(x^2 - y, x + 2y\right)$$

at the point (2, 3). 3

(c) State Fatou's lemma. Give an example of a sequence of measurable function for which strict inequality holds in Fatou's lemma. 3

5. (a) Show that a compact subset of a metric space is bounded. Is the coverse true ? Justify your answer. 3

(b) State Monotone Convergence Theorem. Verify it for the sequence $f_n = X_{[n,n+1]}$, $n = 1, 2, 3, \ldots\ldots\ldots\ldots$ 3

(c) For a sequence of non-negative measurable functions $\{f_n\}$ on a measurable set E, prove that : 4

$$\int_E \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int_E f_n dm$$

6. (a) Show that a metric space X is connected if and only if every continuous function f on X to discrete metric space $\{1, -1\}$ is a constant function. 3

(b) Find the critical points of $f(x, y, z) = x^2y^2 + z^2 + 2x - 4y + 2z$ and classify them. 4

(c) Define the Fourier series for a function $f \in L_1^r[-\pi, \pi]$. Prove that the Fourier series for the function $f(t) = t^2$ on $[-\pi, \pi]$ is $\frac{\pi^2}{3} + 4\sum_{n \in \mathbf{N}} \frac{(-1)^n \cos nt}{n^2}$. 3