No. of Printed Pages : 5 **RMTE-105**

Ph. D. (MATHEMATICS)

(PHDMT)

Term-End Examination

December, 2022

RMTE-105 : PARTIAL DIFFERENTIAL EQUATIONS (ELECTIVE)

Time : 3 Hours *Maximum Marks : 100*

Note : (*i*) *Question No.* ***1*** *is compulsory.*

(*ii*) *Answer any* ***nine*** *questions out of remaining Q. No.* ***2*** *to* ***12****.*

(*iii*) *Use of scientific and non-programmable calculator is allowed.*

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1. State whether the following statements are True *or* False. Justify your answer with the help of a short proof or a counter-example. No marks will be awarded without justification : $2 \times 5 = 10$

(a) Characteristic method can be used to convert some non-linear p.d.e. into a system of o.d.e.

(b) If :

$$\text{L}u = f \text{ in U}$$

$$u = 0 \text{ on } \partial\text{U}$$

where U is an open, bounded subset of $\mathbf{R}^n$ and $u : \overline{\text{U}} \to \mathbf{R}$ is the unknown $u = u(x)$. Here $f : \text{U} \to \mathbf{R}$ is given and

$$\text{L}u = -\sum_{i,\dot{z}=1}^{n} a^{i\dot{z}}(x)\, u_{x_i\, x_{\dot{z}}}$$

$$+\sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)\, u,$$

then the p.d.e. $\text{L}u = f$ is in divergence form.

(c) If X be a vector space, then an inner product on X is a definite positive symmetric, bilinear form on X.

(d) If H = $\mathbf{R}^2$ and $a(x, y) = 2x_1y_1 + 3x_2y_2$, where $x = (x_1, x_2) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}^2$, then $a(x, y)$ is coercive.

(e) Cole-Hopf transformation can be used to solve :

$$u_{tt} - 3u_{xx} + 5u_x^2 = 0 \text{ in } \mathbf{R} \times (0, \infty)$$

$$u = x^2 \text{ on } \mathbf{R} \times \{t = 0\}.$$

2. Determine an explicit solution for a function u solving the initial-value problem : 10

$$u_t + b.\mathrm{D}u = 0 \text{ in } \mathrm{R}^n \times (0, \infty)$$

$$u = x^2 + 1 \text{ on } \mathbf{R}^n \times \{t = 0\}$$

where $b \in \mathbf{R}^n$.

3. Using characteristic method, determine the solution of the equation : 10

$$u_x + 3u_y = u^2 \text{ in U}$$

$$u = g \text{ on } \pi .$$

where U is the half-space $\{x > 0\}$ and $\Gamma = \{x = 0\} = \partial \mathrm{U}$.

4. Use Fourier transformation to solve the equation :

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, t > 0$$

subject to the following conditions : 10

(i) $u, \dfrac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

(ii) $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 0 \end{cases}$

(iii) $u(x, 0) = f(x)$

(iv) $u(x, t)$ is bounded

5. Explain for what type of equation we can apply Cole-Hopf transformation. Using Cole-Hopf transformation, determine the solution of the following equation : 4+6

$$\begin{aligned} u_t - 4u_{xx} + \frac{1}{2}u_x^2 &= 0 \text{ in } \mathbf{R}\times(0,\infty) \\ u &= x + 2 \text{ on } \mathbf{R}\times(t=0) \end{aligned}$$

6. Define integral surface for a semi-linear Cauchy problem. Using Cauchy characteristic method, determine the solution of the equation : 3+7

$$u_x + u_y = u$$

$$u(x,0) = 1 + e^x$$

7. Explain fundamental solution of Laplace equation. 10

8. Solve the following equation by using Laplace transform : 10

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 1 - e^{-t}, \quad 0 < x, t > 0$$

$$u(x,0) = x .$$

9. Explain complete integral and envelope with example. 10

10. Explain and define second order elliptic equation in $\mathbf{R}^n$. Give an example of second order elliptic equation in $\mathbf{R}^3$. 8+2

11. Consider the boundary value problem :

$$Lu + \mu u = f \text{ in } U \subseteq \mathbf{R}^n$$

$$u = 0 \text{ on } \partial U$$

There is a number $\nu \geq 0$ such that $\mu \geq \nu$ and $f \in L^2(U)$. Explain the existence and uniqueness of weak solution for $u \in H_0^1(U)$. 10

12. Explain weak solution of second order parabolic equation. 10