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RMT-101

Ph.D. PROGRAMME IN MATHEMATICS

Term-End Examination

December, 2022

RMT-101 : ALGEBRA

Time : 3 hours *Maximum Marks : 100*

Note : *(i) There are eight questions in this paper.*

*(ii) The eighth question is **compulsory**.*

*(iii) Do any **six** questions from question one to question seven.*

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1. (a) Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) \;\middle|\; ad - bc > 0 \right\}$$

and

$$S = \{z \in \mathbb{C} \mid \mathfrak{I}(z) > 0\}$$

where $\mathfrak{I}(z)$ denotes the imaginary part of a complex number z. Prove that G acts on S by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . z = \frac{az + b}{cz + d}.$$ *7*

(b) Prove that the set of all nilpotent elements in a commutative ring R is an ideal of R. *4*

(c) Let R be a ring with unity that has no proper left ideals. Prove that R is a division ring. *4*

2. (a) Let F be a field. Show that the action of $GL_n(F)$ on $F^n \setminus \{\mathbf{0}\}$ is transitive. If $\mathbb{F}_q$ is the finite field with q elements, find the cardinality of the orbit and the stabiliser of $v = (1, 0, ..., 0)^t \in \mathbb{F}_q^n$ under the action of left multiplication by the elements of $GL_2(\mathbb{F}_q)$. *9*

(b) Show by induction that, *3*

$$|GL_n(\mathbb{F}_q)| = q^{\frac{n(n-1)}{2}} (q^n - 1)(q^{n-1} - 1) \dots (q - 1)$$

(c) If R is a ring and $a \in R$, then $J = \{r \in R \mid ra = 0\}$ is a left ideal of R and $K = \{r \in R \mid ar = 0\}$ is a right ideal of R. *3*

3. (a) If a group G has conjugacy class with two elements, show that G has a proper, non-trivial, normal subgroup. *5*

(b) Let R be a ring with identity and S be the ring of all $n \times n$ matrices over R. J is an ideal of S, if and only if J is the ring of all $n \times n$ matrices over an ideal I of R. *10*

4. (a) Give an example of finite abelian groups G, H_1, H_2, K_1 and K_2 such that $G = H_1 \times H_2$ and $G = K_1 \times K_2$, but no H_i is isomorphic to any K_i. *4*

(b) Let R be a ring with identity.

A matrix $(a_{ij}) \in \text{Mat}_n R$ is said to be

(upper) triangular $\Leftrightarrow a_{ij} = 0$ for $j < i$;

strictly triangular $\Leftrightarrow a_{ij} = 0$ for $j \le i$.

Show that the set T of all triangular matrices is a subring in $\text{Mat}_n R$ and the set I of all strictly triangular matrices is an ideal in T.

Show that $T/I \simeq \underbrace{R \times R \times \dots R}_{n \text{ times}}$. *6*

(c) Let R be the ring of 2×2 matrices over a field F.

(i) Show that the centre of R consists of all the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(ii) Show that the centre of R is not an ideal.

What is the centre of R, if F is not a field, but a division ring ? *5*

5. (a) Find four different subgroups of S_4 which are isomorphic to S_3 and nine different subgroups of S_4 isomorphic to S_2. *5*

(b) Determine all the group homomorphisms $\phi : S_3 \to \mathbb{R}$. *5*

(c) Check whether we have an exact sequence of R-modules

$$0 \to M'' \xrightarrow{f} M \xrightarrow{g} M' \to 0$$

if we take;

(i) $R = \mathbb{Z}$, $M'' = \mathbb{Z}$, $M = \mathbb{Q}$, $M' = \{z \in \mathbb{C} \mid |z| = 1\}$, $f(x) = x$ and $g(x) = e^{2\pi ix}$.

(ii) $R = \mathbb{Z}$, $M'' = (1 - x)\mathbb{Z}[X]$, $M = \mathbb{Z}[X]$, $M' = \mathbb{Z}$, $f(p(X)) = P(X)$ and $g : M \to M'$ given by $g\left(\sum_{i=0}^{n} a_i X^i\right) = \left(\sum_{i=0}^{n} a_i\right)$.

If any of the sequences is exact, check whether it is split exact. If it is split exact find a splitting. If you think it is not split exact, justify your answer. *5*

6. (a) Let G be a group of odd order and let N be a normal subgroup of G with $|N| = 5$. Show that N is contained in the centre of G. *5*

(b) Let $f : A \to A$ be an R-module homomorphism such that $f\,f = f$. Show that $A = \text{Ker } f \oplus \text{Im } f$. *3*

(c) If G is a group and

$$G \simeq Z_6 \oplus Z_{15} \oplus Z_{21} \oplus Z_{25},$$

find the elementary divisors and invariant factors of G. *4*

(d) Show that the additive group Z_{p^n}, $n \in \mathbb{N}$, p a prime, cannot be written as the direct sum of two of its proper subgroups. *3*

7. (a) Find the number of elements of order 5 in a group G of order 20. *3*

(b) Prove that no group of order p^2q, where p and q are distinct primes, is simple, *9*

(c) Suppose R is commutative ring with identity having the property whenever $r + s = 1$ for $r, s \in R$, one of r or s is a unit. Then R is a local ring. *3*

8. Which of the following statements are *True* and which are *False*. Give reasons for your answer. If you think a statement is false, give a counter example. If you think a statement is true give a short proof. *10*

(a) If G is a group of order 11 and S is a set of 7 elements, there is no transitive action of G on S.

(b) Every solvable group is nilpotent.

(c) Every cyclic R-module is simple, where R is a commutative ring with unity.

(d) If A is a submodule of B, then B is Noetherian if A satisfies Ascending Chain Condition on its submodules.

(e) S_3 is the direct product of two of its subgroups.