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MMT-007

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

December, 2022

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 50%)

Note : *Question no.* ***1*** *is* ***compulsory****. Answer any* ***four*** *questions out of the remaining questions no. 2 to 7. Use of scientific non-programmable calculator is allowed.*

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1. State whether the following statements are *True* or *False*. Justify your answer with the help of a short proof or a counter example. *5×2=10*

(a) Lipschitz condition is satisfied for the initial value problem

$$y' = \sqrt{|y|}, \; y(0) = 0$$

on the rectangle $|x| \leq 1$ and $|y| \leq 1$.

(b) If $\mathcal{L}$ is the Laplace transform, then

$$\mathcal{L}\,[t^2 \cos nt] = \frac{2s\,(s^2 - 3n^2)}{s^2 + n^2}.$$

(c) The interval of absolute stability of the two-stage Runge-Kutta method given by

$$y_{i+1} = y_i + \frac{1}{4}\,(k_1 + 3k_2),$$

where $k_1 = h\,f(x_1, y_1)$ and

$$k_2 = h\,f\left(x_1 + \frac{2h}{3}, y_1 + \frac{2}{3}\,k_1\right) \text{ is } -2 < \lambda h < 0.$$

(d) The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4x\,\frac{\partial^2 u}{\partial x\,\partial y} + (1 - y^2)\,\frac{\partial^2 u}{\partial y^2} = 0$$

is elliptic inside the ellipse $4x^2 + y^2 = 1$.

(e) The order of the difference method for hyperbolic equations is $O(k^2 + h^2)$.

2. (a) Find series solution near $x = 0$ of the differential equation

$$(x^2 - x)\,\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} - 2y = 0.$$ *6*

(b) If $\mathcal{L}$ is the Laplace operator, then show that

$$\mathcal{L}\,\{t^n\} = \frac{\sqrt{(n+1)}}{s^{n+1}},\ n > -1,\ s > 0.$$ *2*

(c) Using Convolution theorem, evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}.$$ 2

3. (a) Solve the initial value problem

$$y' = x + y^2,\ y(0) = 1,$$

on the interval [0, 0·4] using the Runge-Kutta second order method with h = 0·2. 5

(b) Using Laplace transform technique, solve the differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2},\ x > 0,\ t > 0$$

given that y(0, t) = 10 sin 2t, y(x, 0) = 0,

$$\frac{\partial y}{\partial t}(x, 0) = 0, \quad \lim_{x \to \infty} y(x, t) = 0$$ 5

4. (a) Solve the boundary value problem

$$\frac{\partial^2 y}{\partial x^2} = 2y\,\frac{\partial y}{\partial x},\quad y(0) = \frac{1}{2},\quad y(1) = 1$$

using second order finite difference method with $h = \frac{1}{3}$. 5

(b) Using Milne's fourth order predictor-corrector method, find y(2), given

$$\frac{dy}{dx} = \frac{1}{2}(x + y),\ y(0) = 2,$$

where y(0·5) = 2·636, y(1) = 3·595, y(1·5) = 4·968. Perform two corrector iterations. *5*

5. (a) The wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is approximated by

$$\delta_t^2\, u_i^n = \frac{s^2}{2}\, \delta_x^2 \left[u_i^{n+1} + u_i^{n-1}\right],$$

where $s = \frac{k}{h}$. Investigate the stability using Von Neumann method. *5*

(b) Solve two-dimensional Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,\ \ x > 0,\ \ 0 < y < b$$

subject to the conditions

$v(x, 0) = f(x),\ \ x > 0$

$v(x, b) = 0,\ \ x > 0$

$v(0, y) = 0,\ \ 0 < y < b,$

using Fourier sine transforms. *5*

6. (a) Using the generating function for $J_n(x)$, prove that $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$, for integer values of n. 5

(b) Show that

$$H_{2n+1}(0) = 0 \text{ and}$$

$$H'_{2n+1}(0) = (-1)^n \frac{\lfloor 2n+2}{\lfloor n+1}.$$ 5

7. (a) Solve the following boundary value problem, by determining the appropriate Green's function by using the method of variation of parameters :

$$y'' + y + f(x) = 0, \quad y(0) = 0, \quad y(1) = 0$$

Express the solution as a definite integral. 5

(b) Using five-point formula, find the solution of $\nabla^2 u = 0$ in the square $0 \le x \le 1,\ 0 \le y \le 1$ subject to the boundary conditions

$$u(x, y) = x + y \quad \text{on} \quad x = 0, y = 0, y = 1,$$

$$u + \frac{\partial u}{\partial x} = 1 + x + y \quad \text{on} \quad x = 1.$$

Use the central difference approximation in the boundary condition. Also assume uniform step length $h = \frac{1}{2}$ along the axes. 5