No. of Printed Pages : 3

MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

December, 2022

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *of the remaining questions.*

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1. (a) For x, y, z in an inner product space X, prove that

$$\| x - y \|^2 + \| x - z \|^2 = 2 \left(\| x - \frac{(y + z)}{2} \|^2 + \| y - \frac{(y + z)}{2} \|^2\right).$$

What is the geometric meaning of the identity ? *2+2=4*

(b) Give an example of an operator A such that $O \in \sigma(A)$, but O is not an eigenvalue of A. *3*

(c) Find a bounded linear functional on C that vanishes on C_0. *3*

2. (a) Let X be a normed linear space and $a \in X, a \neq 0$. Show that
$\| a \| = \sup \{ | f(a) | : f \in X^1, \| f \| \leq 1 \}$. *4*

(b) Find a discontinuous linear functional on l^1. *3*

(c) If H, K are Hilbert spaces, show how to make $H \times K$ into a Hilbert space. *3*

3. (a) Let $M = \text{span} \{1, x\} \subset C[-1, 1]$. Prove that $Q(a + bx) = a$ defines a bounded linear functional on M and calculate $\| Q \|$. Define $\psi_1(f) = f(0), f \in C[-1, 1)$. Show that ψ_1 is a Hahn-Banach extension of Q to $C[-1, 1]$. *2+2+2=6*

(b) Find an orthonormal basis of $\mathbf{R}^3$ containing $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$. *2*

(c) If $x \in l^1$, show that $\sum x(n)e_n$ converges to x in norm. *2*

4. (a) Let X be a Banach space and let B ($\mathbf{R}$, X) be the space of all bounded maps from $\mathbf{R}$ to X. Prove that B($\mathbf{R}$, X) is a Banach space with norm $\| f \| = \sup \{ \| f(t) \|_X : t \in \mathbf{R} \}$. *5*

(b) State the Riesz Representation theorem for Hilbert spaces. Use it to prove the Hahn-Banach extension theorem for Hilbert spaces. *1+4=5*

5\. (a) Let A be a bounded linear operator on a Hilbert space H satisfying $\langle Ax, x \rangle \geq 0$ for all $x \in H$. If $\lambda \in \sigma(A)$, show that $\lambda \geq 0$. Give an example. *2+2=4*

(b) Find a bounded linear functional Q on C[0, 1] with $\| Q \| = e$. *3*

(c) Let M & N be closed subspaces of a Hilbert space H. If $M \perp N$, prove that M + N is closed. *3*

6\. State whether the following statements are *True* or *False*. Justify your answers. *5×2=10*

(a) Every 1-dimensional Banach space is a Hilbert space.

(b) If an operator on l^2 is bounded below, it is not compact.

(c) $l^3 \subset l^2$.

(d) If A, B are self-adjoint operators, so is AB.

(e) There is a linear functional ϕ on l^∞ with $\| \phi \| = 1$ such that $\phi(e_n) = 0$ for all n.