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MMT-003

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

December, 2022

MMT-003 : ALGEBRA

Time : 2 hours *Maximum Marks : 50*

Note : *Question no.* ***1*** *is* ***compulsory****. Answer any* ***four*** *questions from Questions No. 2 to 6. The use of calculators is* ***not*** *allowed.*

1. State whether the following statements are *True* or *False*, giving reasons for the answers : *10*

(a) A field with 2^{60} elements must have a subfield with 2^{40} elements.

(b) S_{10} has an element of order 30.

(c) There exists a non-abelian group of order 35.

(d) $\mathbf{Z} \times \mathbf{Z}$ is a free group.

(e) If K is a finite field and F a proper subfield, K must be separable over F.

2. (a) Define the conjugacy class of an element in a group. If G is a finite group, write down its class equation carefully explaining each term in the equation. (You need not prove the class equation). *3*

(b) For a finitely generated group G, define the Betti number and the invariant factors of G. What is the Betti number and the invariant factors of $\mathbf{Z}^3 \times \mathbf{Z}_6 \times \mathbf{Z}_8 \times \mathbf{Z}_9$? *3*

(c) State the quadratic reciprocity law for odd primes p and q. Use the quadratic reciprocity law to find $\left(\frac{19}{37}\right)$. *2*

(d) Define a complete set of residues modulo n. Check whether {–1, 0, 1} is a complete set of residues modulo 3. *2*

3. (a) If L/K is an extension of fields, when is $\alpha \in L$ algebraic over K ? Is $1 + \sqrt[3]{2} \in \mathbf{R}$ algebraic over $\mathbf{Q}$? Justify your answer. *3*

(b) Define a separable polynomial over a field F. *2*

(c) Define a symplectic matrix. Give an example of a 2×2 symplectic matrix different from the identity. *2*

(d) Define each of the following : *3*

(ii) Free group on a set S.

(iii) Defining relations of a group G with respect to a generating set X of G.

4. (a) How many non-isomorphic classes of abelian groups can there be of order 120 ? Write down a list of such groups (one from each class). How many of these groups are cyclic ? *4*

(b) Find the splitting field of $x^4 - 3$ over $\mathbf{Q}$. Also find $[K : \mathbf{Q}]$. Further, does K contain a subfield which is not normal over $\mathbf{Q}$? Give reasons for your answer. *6*

5. (a) Let $r : \mathbf{Z} \to \mathbf{Z}_4 \times \mathbf{Z}_7 \times \mathbf{Z}_9$ be the natural homomorphism defined by r(x) = (x(mod 4), x(mod 7), x(mod 9)). Find a pre-image of (2(mod 4), 3(mod 7), 5(mod 91)) lying in [50, 200]. *4*

(b) If p is a prime, show that the centre of a p-group has order p^m for some $m \in \mathbf{N}$. Deduce that every group of order p^2 is abelian. *6*

6. (a) Let G be a group of order 15 acting on a set S with $|S| = 14$. Suppose no element of S is fixed by the entire group G. What are the possible class equations for the action of G on S ? Justify your answer. *4*

(b) Determine whether the polynomial $X^3 + \bar{2}X + \bar{1}$ is irreducible in $\mathbf{Z}_5[X]$. From your answer determine the structure of the ring $\dfrac{Z_5[X]}{\langle X^3 + \bar{2}X + \bar{1} \rangle}$. Is the ring finite ? What is its order ? *4*

(c) What is the order of (2, 3) in $\mathbf{Z}_6 \times \mathbf{Z}_6$? *2*