No. of Printed Pages : 10 **MMT-008**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

[M. Sc. MACS)]

Term-End Examination

December, 2022

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Note : (*i*) *Question No. 8 is compulsory.*

(*ii*) *Attempt any **six** questions from Question Nos. **1** to **7**.*

(*iii*) *Use of scientific (non-programmable) calculator is allowed.*

(*iv*) *Symbols have their usual meanings.*

1. (a) Consider the Markov chain with the following transition probability matrix : 8

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \end{array}$$

(i) Draw the diagram of a Markov chain.

(ii) Classify the states of a Markov chain i.e. persistent, transient, non-null and a periodic state. Also, check the irreducibility of a Markov chain.

(iii) Find the closed set.

(iv) Find the probability of absorption to the class.

(v) Find the mean first passage times from states 1, 2, 3 to state 0.

(b) Find the principal component and proportion of total population variance explained by each component when the covariance matrix is : 7

$$\Sigma = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

2. (a) One bag contains three identical cards marked 1, 2, 3 and another contains two cards marked 1, 2. Two cards are randomly chosen, one from each bag and the numbers observed denote the minimum of the two by X and sum of the two by Y. Find the joint probability distribution of the random pair (X, Y). Find also (i) the marginal distributions of X and Y, (ii) E (X) and E (Y) and (iii) Cov (X, Y). 7

(b) A bank has two tellers working on saving accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that service time distributions of both deposits and withdrawals are exponential with a mean service time of 3 minutes per customer. Depositors and withdrawers arrive in a Poisson fashion throughout the day with 16 and 14 per hour respectively. What would be the effect on the average waiting time for both if each teller could handle both withdrawals and deposits ? What would be the effect if this could only be accomplished by increasing the service time to 3.5 minutes ? 8

3. (a) In a sequence of Bernoulli trials with the probability of success p say that at time n the state E_1 is observed if the trials number $(n - 1)$ and n resulted in SS. Similarly E_2, E_3, E_4 stands for SF, FS, FF. Find the transition probability matrix P of a Markov chain. Also find $P^m (m \geq 2)$ and its asymptotic behaviour. 7

(b) Two samples of size 20 and 30 were taken from two socio-economic status by two methods 1 and 2. Two characteristics X_1 and X_2 were measured. The summary statistics for the socio-economic status is given by : 8

$$\bar{X}_1 = \begin{bmatrix} 9 \\ 5 \end{bmatrix}, \qquad S_1 = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\bar{X}_2 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Test at 5% level of significance whether $\mu_1 = \mu_2$ or not by stating the assumptions. [You may like to use the values : $F_{2,47(0.05)} = 3.20$, $F_{2,49(0.05)} = 4.05$, $F_{2,50\ (0.50)} = 4.50$].

4. (a) For a linear growth process with $\lambda_n = n\lambda, \mu_n = n\mu, \lambda_0 = \mu_0 = 0$ and the initial population size is i at $t = 0$, find mean and variance of population, size $x(t)$ at time t; when $\lambda \neq \mu$. 9

(b) Find the values of a, b and c for which the following matrix A will be orthogonal : 6

$$A = \begin{pmatrix} \frac{a}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{b}{3} \\ \frac{2}{3} & \frac{c}{3} & \frac{1}{3} \end{pmatrix}$$

5. (a) The following data, gives DPS (dividend per share), EPS (earning per share) and return on market for six companies : 9

DPS (Y)	EPS (X_1)	Return on Market (X_2)
1	3	2
2	5	3
3	5	4
2	4	5
4	8	6
5	7	8

Find :

(i) The regression equation $Y = b_0 + b_1 X_1 + b_2 X_2$.

(ii) Multiple correlation co-efficient.

(b) If the offspring distribution in a branching process is such that : 6

$$p_n = \left(\frac{1}{2}\right)^{n+1}, \quad n = 0, 1, 2 \ldots\ldots\ldots$$

then find the p.g.f. of nth generation in a branching process. Also, find the probability of ultimate extinction.

6. (a) Suppose the life-times $X_1, X_2, \ldots\ldots\ldots$ are i. i. d. uniformly distributed on [0, 2] and $c_1 = 2$, $c_2 = 8$. Let T > 0 and age replacement policy is to be employed : 7

   (i) Find μ^T.

   (ii) Find the long-run average cost per unit time.

   (iii) In this case which is a better policy in the long-run in terms of cost–

   Individual replacement policy or age replacement policy ?

(b) Let the data matrix for a random sample of size $n = 3$ from a bivariate normal population be : 8

$$X = \begin{bmatrix} 4 & 6 \\ 8 & 4 \\ 6 & 2 \end{bmatrix}$$

Evaluate the observed T^2 for $\mu_0^1 = (7, 3)$. Find the sampling distribution of T^2.

7. (a) Suppose that $n_1 = 15$ and $n_2 = 20$ observations are made on two variables X_1 and X_2. Given : 7

$$\mu^{(1)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \mu^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \sum = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

Considering equal cost and equal prior probabilities, check whether the observation (3.5, 2) belongs to π_1 or π_2.

(b) Let $X \sim N_2(\mu, \Sigma)$, where $\mu = (4, 3, 5)'$ and

$$\sum = \begin{bmatrix} 6 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}.$$

Find the distribution of $\begin{bmatrix} 2X_1 + X_2 - X_3 \\ X_1 - X_2 + 2X_3 \end{bmatrix}$. 8

8. State which of the following statements are true *or* false. Give a short proof or counter-example in support of your answer : 10

   (i) The interval between two successive occurrences of a Poisson process having parameter λ_1 has a negative exponential distribution with mean $\frac{1}{\lambda_1}$. Similarly for another independent Poisson process with parameter λ_2 has an exponential distribution with mean $\frac{1}{\lambda_2}$. Then the interval between two successive occurrences of the above two combined Poisson process has exponential distribution with mean $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$.

   (ii) For two random variables X and Y :

   $$V(X) = E(V(X)/Y) - (E(X/Y))^2$$

   (iii) Every discrete parameter and discrete state space stochastic process is a Markov process.

(iv) For M/M/I model, the average length of non-empty queue is $\frac{\mu}{\lambda(\mu-\lambda)}$.

(v) Let :

$$\underset{\sim}{X}_j^1 = (x_{j_1}, x_{j_2}, \ldots\ldots, x_{jk})$$

then $\sum_{i=1}^{n}(X_j - \bar{X})(\bar{X} - \mu)$ is a (k × k) matrix of zeroes, where $\underset{\sim}{X}_j \sim N_k(\mu, \Sigma)$.