No. of Printed Pages : 6 **MMT-004**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

December, 2022

MMT-004 : REAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

***Note* :** *(i) Question No. **1** is compulsory.*

*(ii) Attempt any **four** questions from Q. Nos. 2 to 6.*

(iii) Calculator is not allowed.

(iv) Notations as in the study material.

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1. State whether the following statements are True *or* False. Justify your answers : 5×2=10

(a) The sequence $\left\{\left(\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ in $\mathbb{R}^2$ under the discrete metric on $\mathbb{R}^2$ converges in $\mathbb{R}^2$.

(b) A subset in a metric space is compact if it is closed.

(c) Continuous image of a path connected space is path connected.

(d) The second derivative of a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ never vanishes.

(e) If $\int_A f\, dm = \int_A g\, dm$ for all $A \in M$, then $f = g$.

2. (a) Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous at a point $c \in X$ if and only if given a closed set V containing $f(c)$ in Y, we can find a closed set u containing c such that $f(u) \subset V$. 3

(b) Define directional derivatives. If $f(x, y, z, w) = (x^2 - y^2, 2xy, zx, x^2z^2w^2)$ and $v = (2, 1, -2, 0)$, find $f'(1, 2, -1, -2)$ and $D_v(1, 2, -1, -2)$. 3

(c) Define a Lebesgue measurable function. Prove that a continuous real function defined on a measurable subset of $\mathbb{R}$ is measurable. Is a measurable function continuous ? Justify. 4

3. (a) State and prove the Gluing lemma for a finite family of closed sets. 4

(b) Making the usual assumptions, define the partial derivatives of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. For the function $F : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $F(x, y, z, w) = (x^2y, xyz, x^2 + y^2 + zw^2)$, find $F'(a)$, where $a = (1, 0, -1, 0)$. 3

(c) When is a non-negative measurable function defined on a measurable set of $\mathbb{R}$ said to be Lebesgue integrable ? Find the Lebesgue integral of the function f defined by : 3

$$f(x) = \begin{cases} 6, & x \in [1, 2] \\ 1, & x \in (2, 4) \\ 0, & \text{elsewhere} \end{cases}$$

4. (a) Define completeness in a metric space. Give an example of a metric space which is not complete. Justify your choice of example. 3

(b) State the Inverse function theorem. Using the theorem, prove that if f is a C^1 function defined on an open set $E \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$ with $Jf(x) \neq 0$ for all $x \in E$, then the image of $f(V)$ of any open set VCE is an open set in $\mathbb{R}^n$. 4

(c) State Fatou's lemma and use it to prove the monotone convergence theorem. 3

5. (a) Prove that the continuous image of a connected set is connected in a metric space. 3

(b) Find the extreme values of the function

$$Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 = 50$$

subject to the constraint, $x_1 + x_2 + x_3 = 20$, $x_1, x_2, x_3 \geq 0$. 3

(c) Find the Fourier series for the function $f(t) = t^2$ on $[-\pi, \pi]$. 4

6. (a) (i) Define a linear system. 1

(ii) Let h be a scalar valued function. Let $\mathbb{R} : S \to S$ be the system given by :

$$\mathbb{R}\, f(t) = \int_{-}^{\infty} h(\tau)\, f(t - \tau)\, d\tau$$

Prove that the system $\mathbb{R}$ is a linear system, where S is the set of signals. 2

(b) For a function $f \in L^1(\mathbb{R})$ define its Fourier transform $\hat{f}$ and prove that $\hat{f}$ is continuous on $\mathbb{R}$. Prove also that for $f, g \in L^1(\mathbb{R})$, $(\hat{f} * g)(w) = f(w)g(w)$. 5

(c) Find the interior and closure of the set $A = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$ as a subset of $\mathbb{R}^2$ with standard metric. 2