No. of Printed Pages : 7

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

December, 2021

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours | *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no.* ***8*** *is* ***compulsory****. Attempt any* ***six*** *questions from question nos. 1 to 7. Use of scientific non-programmable calculator is allowed. Symbols have their usual meaning.*

---

1. (a) One bag contains three identical cards marked 1, 2, 3 and another bag contains 2 cards marked 1, 2. Two cards are randomly chosen, one from each bag and the numbers are observed. Let X denote the minimum of two and Y denote the sum of two. Find the joint probability distribution of the random variable pair (X, Y). Also find the marginal distributions of both X and Y, conditional distribution of $(X \mid Y \leq 4)$, $E(X \mid Y = 4)$ and $V(X \mid Y = 4)$. *9*

(b) Let $X \sim N_3(0, \Sigma)$, where $\Sigma = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 5 & -3 \\ 3 & -3 & 5 \end{pmatrix}$.

(i) Find the correlation matrix.

(ii) Find the distribution of $X_1 + 2X_2 - 3X_3$. *6*

2. (a) Find the principal components and the proportion of total population variance for the first principal component, when covariance matrix is

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}; \; -\frac{1}{\sqrt{2}} < \rho < \frac{1}{\sqrt{2}}.$$ *9*

(b) If $g(\lambda)$ is a gamma density with parameters m and θ, then show that

$$P[N(t) = n] = {}^{m+n-1}C_n \left(\frac{\theta}{\theta + t}\right)^m \left(\frac{t}{\theta + t}\right)^n.$$ *6*

3. (a) Past records indicate that of a large number of machines that a factory owns, breakdowns occur at random and the average time between the breakdowns is 2 days. Assuming that the repairing capacity of the workman is one machine a day and the repairing time is distributed exponentially, determine the following :

(i) Identify the Model.

(ii) The probability that the service facility will be idle.

(iii) The expected length of queue (L_q).

(iv) The expected number of machines waiting to be repaired and being repaired (L_s).

(v) The expected time that a machine shall wait in the queue to be repaired (W_q).

(vi) The expected time a machine shall be idle for reasons of waiting to be repaired and being repaired (W_s). *9*

(b) Guns 1 and 2 are shooting at the same target. Gun 1 shoots on an average 9 shots during the same time gun 2 shoots 10 shots. The precision of these two guns is not the same. On an average, out of 10 shots from gun 1, eight hit the target and from gun 2, only seven hit the target. In the course of shooting, the target has been hit by a bullet, but it is not known which gun shot the bullet. Find the chance that the target was hit by gun 2. *6*

4. (a) Let the hypothetical data matrix for a random sample of size n = 3 from bivariate normal population be

$$X = \begin{pmatrix} 8 & 5 \\ 5 & 9 \\ 2 & 7 \end{pmatrix}.$$

Evaluate the observed T^2 for $\mu_0' = (4,\ 8)$ and find the sampling distribution of T^2. *6*

(b) Describe the birth and death process with parameter λ_n and μ_n, where λ_n is the birth rate when population size is n at time t and μ_n is the death rate at time t.

If $\lambda_n = \dfrac{\lambda}{n+1}$ for $n \geq 0$ and $\mu_n = \mu$ for $n > 1$, then find the stationary distribution and show that it always exists. *9*

5. (a) Random variable (X, Y) has the joint bivariate normal distribution with p.d.f.

$$f(x, y) = \frac{1}{\left(6\pi\sqrt{3}\right)}$$

$$\exp\left[-\frac{2}{3}\left\{\frac{1}{9}x^2 - \frac{5}{6}x + \frac{1}{6}xy + \frac{1}{4}y^2 - y + \frac{7}{8}\right\}\right]$$

Find all the five parameters and express it in a standard form. *8*

(b) To fit a linear regression of dependent variable Y on independent variables X_1 and X_2 with the following information : $E(Y) = 5$, $E(X_1) = 2$, $E(X_2) = 4$, $V(X_1) = 1$, $V(X_2) = 4$, $V(Y) = 9$, $Cov(X_1, X_2) = -0 \cdot 4$, $Cov(X_1, Y) = 1 \cdot 2$ and $Cov(X_2, Y) = 1 \cdot 8$. Fit a linear regression of the form

$Y = b_0 + b_1X_1 + b_2X_2$. Also find the multiple correlation coefficient. $R_{Y \cdot X_1X_2}$ *7*

6. (a) The transition probability matrix of a Markov chain $\{X_n;\ n = 1, 2, ...\}$ having 3 states 1, 2 and 3 is

$$P = \begin{pmatrix} 0 \cdot 1 & 0 \cdot 5 & 0 \cdot 4 \\ 0 \cdot 6 & 0 \cdot 2 & 0 \cdot 2 \\ 0 \cdot 3 & 0 \cdot 4 & 0 \cdot 3 \end{pmatrix}$$

and the initial distribution is

$$\pi_0 = (0 \cdot 7, 0 \cdot 2, 0 \cdot 1).$$

Find : *9*

(i) $P(X_1 = 3)$

(ii) $P(X_1 = 3, X_2 = 2)$

(iii) $P(X_0 = 2, X_1 = 3, X_2 = 2, X_3 = 3)$

(b) For a branching process having

$p_0 = \frac{1}{6}$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{3}$, calculate the probability of ultimate extinction.

Also explain what is ultimate extinction. *6*

7. (a) Compute a lower triangular square root of a matrix $A = \begin{pmatrix} 9 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{pmatrix}$. *5*

(b) Consider the following transition probability matrix of a Markov chain :

$$P = \begin{pmatrix} 0{\cdot}3 & 0{\cdot}7 & 0 & 0 & 0 \\ 0{\cdot}5 & 0{\cdot}5 & 0 & 0 & 0 \\ 0 & 0 & 0{\cdot}8 & 0{\cdot}2 & 0 \\ 0 & 0 & 0{\cdot}4 & 0{\cdot}3 & 0{\cdot}3 \\ 0 & 0 & 0 & 0{\cdot}1 & 0{\cdot}9 \end{pmatrix}$$

Calculate the following : *10*

(i) Represent the Markov chain by directed graph.

(ii) Classify the different types of states : States which communicate, recurrent state, transient state, absorbing state.

(iii) Irreducible chain.

(iv) If the state is recurrent, then find mean recurrence time and its periodicity.

8. State whether the following statements are *true* or *false*. Justify your answer with a short proof or a counter-example : *10*

(a) If P is a transition probability matrix of a Markov chain, then all column and all row sums are equal to unity.

(b) If X_1, X_2, X_3 are i.l.d. from $N(\mu, \Sigma)$, then $\frac{X_1 + X_2}{2}$, $\frac{X_2 + X_3}{2}$ and $\frac{X_3 + X_1}{2}$ are also i.l.d. from $N(\mu, \Sigma)$.

(c) If the arrival distribution in a queueing model $M|M|1|\infty|FIFO$ is Poisson, then the service distribution is also Poisson.

(d) Poisson process is an example of Renewal process.

(e) Level of significance and critical region are related by P(|test statistics|< corresponding tabulated value) = α

i.e., $P\left[\chi^2 < \chi^2_{\alpha, n}\right] = \alpha$.