No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

February, 2021

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours | *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no.* ***8*** *is* ***compulsory****. Answer any* ***six*** *questions from question nos. 1 to 7. Use of non-programmable calculator is* ***allowed****. All the symbols used have their usual meaning.*

1. (a) Let the joint probability density function of two discrete random variables X and Y be given as : *8*

		X			
		2	3	4	5
Y	0	0	0·03	0	0
	1	0·34	0·30	0·16	0
	2	0	0	0·03	0·14

(i) Find the marginal distribution of X and Y.

(ii) Find the conditional distribution of X given Y = 1.

(iii) Test the independence of variables X and Y.

(iv) Find $V\left[\frac{Y}{X} = x\right]$.

(b) Let $p_k = bc^{k-1}$, $k = 1, 2, \ldots, 0 < b, c,\ b + c < 1$,

and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$.

Find the probability of ultimate extinction. 7

2. (a) A computer lab has a help desk to assist students working on computer spreadsheet assignments. The students form a single line and are served on a first-come first-served basis. On average, 15 students per hour arrive in a Poisson distribution. The help desk can help an average of 20 students per hour and the service rate is exponentially distributed.

Find

(i) The average utilization of the help desk.

(ii) The average number of students in the system.

(iii) The average number of students waiting in line.

(iv) The average time a student spends in the system.

(v) The average time a student spends waiting in line.

(vi) The probability of having more than 4 students in the system. 7

(b) Let $\mathbf{X} \sim N_3(\mu, \Sigma)$, where $\mu = [1 \quad 2 \quad 3]'$ and

$$\Sigma = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find the distribution of

$$\begin{bmatrix} 2X_1 + X_2 + X_3 \\ X_1 - X_2 + X_3 \end{bmatrix}.$$

Also, find the conditional distribution of X_1 given $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$. *8*

3. (a) Consider the Markov chain with three states, $S = \{1, 2, 3\}$ following the transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}.$$

(i) Draw the state transition diagram for this chain.

(ii) If $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, then find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

(iii) Check whether the chain is irreducible and aperiodic.

(iv) Find the stationary distribution for the chain. *9*

(b) Determine the principal components Y_1 and Y_2 for the covariance matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Also, calculate the proportion of total population variance for the first principal component. *6*

4. (a) If $N_1(t)$, $N_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then show that

$$P[N_1(t) = k \mid N_1(t) + N_2(t) = n] = {}^{n}C_k \, p^k \, q^{n-k},$$

where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$, $q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$. *4*

(b) Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the mean vector $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Suppose $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{Y} \sim N_3.$$

(i) Find $P(0 \le X_2 \le 1)$.

(ii) Compute $E(\mathbf{Y})$.

(iii) Find the covariance matrix of $\mathbf{Y}$.

(iv) Find $P(Y_3 \le 4)$. *7*

(c) A box contains two coins : a regular coin and one fake two-headed coin. One coin is chosen at random and tossed twice. The following events are defined :

A : first coin toss results in a head.

B : second coin toss results in a head.

C : coin 1 (regular) has been selected.

Find $P(A \mid C)$, $P(B \mid C)$, $P(A \cap B \mid C)$, $P(A)$, $P(B)$ and $P(A \cap B)$. *4*

5. (a) Let the mean vectors and covariance matrices of the variables $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $\mathbf{Y}$ are $\boldsymbol{\mu}_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{\mu}_Y = 3$, $\Sigma_{XX} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $\boldsymbol{\sigma}_{YY} = 14$ and $\boldsymbol{\sigma}_{XY} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) Fit the equation $\mathbf{Y} = b_0 + b_1 X_1 + b_2 X_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean squared error. *6*

(b) Suppose the interoccurrence times $\{X_n : n \geq 1\}$ are uniformly distributed on $[0, 1]$. Find the Laplace transform of the renewal function M_t. Also, find $\lim\limits_{t \to \infty} \dfrac{M_t}{t}$. *4*

(c) Consider three random variables X_1, X_2, X_3 having the covariance matrix $\begin{bmatrix} 1 & 0{\cdot}12 & 0{\cdot}08 \\ 0{\cdot}12 & 1 & 0{\cdot}06 \\ 0{\cdot}08 & 0{\cdot}06 & 1 \end{bmatrix}$. Write the factor model, if number of variables and number of factors are 3 and 1 respectively. *5*

6. (a) Two samples of sizes 40 and 60, respectively, were drawn from two different lots of a certain manufactured component. Two characteristics X_1 and X_2 were measured for the sampled items. The summary statistics of the measurements for lots 1 and 2 is given below :

$$\overline{X}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \ \overline{X}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \ S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Assume that X_1 and X_2 are in normal distribution and $\Sigma_1 = \Sigma_2$. Test at 99% level of significance whether $\mu_1 = \mu_2$ or not. [You may like to use the values $F_{2, 97}(0{\cdot}01) = 4{\cdot}86$, $F_{3, 97}(0{\cdot}01) = 4{\cdot}05$, $F_{4, 97}(0{\cdot}01) = 3{\cdot}89$] *9*

(b) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval [2, 5] years. Further, planned replacements take place every 3 years.

Compute

(i) long-term rate of replacements.

(ii) long-term rate of failures. *6*

7. (a) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2, where $X_1 \sim N_2(\mu_1, \Sigma)$ and $X_2 \sim N_2(\mu_2, \Sigma)$. Given is $\mu_1 = [1 \quad 2]'$, $\mu_2 = [-1 \quad 0]'$ and $\Sigma = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}'$.

Considering equal cost and equal prior probabilities, classify the observation $[-1 \quad 1]'$ in one of the two populations. *7*

(b) One of the two teller machines handles withdrawals only, while the other handles deposits only in a bank. The service time of both the machines follows exponential distribution with mean service time 3 minutes. The depositors arrive in the bank at the rate of 16 per hour and withdrawers arrive at the rate of 14 per hour in Poisson distribution. Find the average waiting times of depositors and withdrawers in the queue. If each machine can handle both the jobs of deposits and withdrawals, then what will be the average waiting time in the queue for a customer ? *8*

8. State whether the following statements are ***true*** or ***false***. Give a short proof or a counter-example in support of your answer. *10*

(i) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is a variance-covariance matrix of two-dimensional random variable.

(ii) The quadratic form $x_1^2 - x_2^2$ is positive definite.

(iii) If a subset of state space of a Markov chain is closed, then any state of that subset can communicate with a state outside the state subset.

(iv) Principal components depend on the scales used to measure the variables.

(v) If X and Y are two random variables with $V(X) = V(Y) = 2$, then $-2 < \text{cov}(X, Y) < 2$.