No. of Printed Pages : 5 **MMT-006**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M. Sc. (MACS)

Term-End Examination

December, 2020

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question* ***No. 6*** *is* ***compulsory****.*

(ii) Answer any ***four*** *questions out of Q. Nos.* ***1*** *to* ***5****.*

(iii) Notations are same as in the study material.

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1. (a) Define the spectral radius of a bounded linear operator acting on a normed space. Let $X = \mathbf{R}^3$ and $A \in B(X)$ be given by the matrix :

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $\sigma(A)$ and $\gamma_\sigma(A)$. 3

(b) Define : 4

$T : C'[0,1] \to C[0,1]$ as $T(f) = f'$.

(i) If $C'[0,1]$ is equipped with $\|.\|_\infty$ norm, then show that T is not bounded.

(ii) If C′ [0, 1] is equipped with $\|.\|'$ norm given by :

$$\|f\|' = \|f\|_\infty + \|f'\|_\infty$$

Then show that T is bounded.

(c) Let u_n be the sequence in l^2 with 1 in the nth place and zeros elsewhere. Prove that the set $\{u_n\}$ is an orthonormal basis for l^2. 3

2. (a) Let X be a Banach space and Y be any normed space and F : X → Y be linear with the following property : If $\{x_n\}$ is a sequence in X such that $x_n \to 0$ and $\{F(x_n)\}$ is Cauchy, then $F(x_n) \to 0$. Prove that F is continuous. 4

(b) Let X = l^2, for n = 1, 2,....... Let x_n = (1, 1, 1, ...1, 0, 0, ...), where 1 occurs only in the first n-places. Orthonormalise this set using the Gram-Schmidt orthonormalization process under the standard inner product on l^2. 3

(c) Let H be a Hilbert space and A $\in$ BL(H). Prove that R (A*) = A if y and only y' A is bounded below. 3

3\. (a) If H is an infinite dimensional Hilbert space, then prove that every isometric isomorphism of H onto itself is unitary. 3

(b) Prove that the unit ball of a normed linear space is compact if and only if the normed linear space is finite dimensional. Use this to show that the identity map on an infinite dimensional normed space is not compact. 4

(c) Let X be a normed space, Z $\in$ X and $f \in \mathrm{X}'$. Let T : X $\to$ X be defined by T (x) = $f(x)z, x \in \mathrm{X}$. Prove that T is linear and compact. 3

4. (a) Show that any ***two*** norms on a finite dimensional normed linear spaces are equivalent. 3

(b) Let Y be a closed subspace of a normed space X over the reals. Let $x_0 \in X$ such that $x_0 \notin Y$. Let $d = d(x_0, Y)$, distance of x_0 from Y. Prove that there exists a functional $f_0 \in X'$ such that $f_0(Y) = 0, \|f_0\| = 1$ and $f_0(x_0) = d$. 4

(c) Let H be a Hilbert space and A be a positive operator on H. Prove that I + A is invertible in BL (H). 3

5. (a) Prove that a Banach space is relfexive if and only if its dual space is reflexive. 4

(b) Let X be a Banach space and let $P : X \to X$ be a projection. If the kernel and range of P are closed subspaces of X, then show that P is continuous. 3

(c) Consider the linear map :

$$T : C[0,1] \to C[0,1]$$

given by $Tf(t) = \int_0^t f(s)\,ds$.

Show that T is bounded. Also calculate $\|T\|$. 3

6. Are the following statements True or False ? Justify your answers with a short proof or a counter example : 5×2=10

   (a) $L'[a,b]$ is a reflexive space.

   (b) Every Banach space is also a Hilbert space.

   (c) There exists an infinite dimensional Banach space X such that every proper subspace of X is complete.

   (d) If the dual X' of a normed linear space X is finite dimensional, then X is finite dimensional.

   (e) If $T : X \to Y$ is continuous, linear and open, then T is surjective.