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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00911

Term-End Examination

December, 2017

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *of the remaining questions.*

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1. (a) Let $||\cdot||_1, ||\cdot||_2$ be norms on a linear space X and let $\varepsilon > \theta$, $\delta > 0$ be fixed. Find conditions on the norms so that $B_1(0, \varepsilon) \subset B_2(0, \delta)$ holds. *3*

   (b) If $||\cdot||_1, ||\cdot||_2$ are equivalent norms on a linear space X and if $(X, ||\cdot||_1)$ is complete, prove that $(X, ||\cdot||_2)$ is also complete. *3*

   (c) State the projection theorem and illustrate it on $L^2[0, 1]$ with $M = \{f \in L^2[0, 1] : f = 0$ a.e. on $[0, 1/2]\}$. *1+3*

2. (a) Show that $(C_{00}, \|\cdot\|_\infty)$ is not complete. *3*

(b) State the open mapping theorem and deduce the closed graph theorem. *1+3*

(c) If A is a bounded linear operator on a Hilbert space H, prove that $R(A^*) = H$, if and only if A is bounded below. *3*

3. (a) Calculate the norm of the linear functional $f \mapsto \int_0^1 t\, f(t)\, dt$ on each of the spaces $(C[0,1], \|\cdot\|_\infty)$ and $(L'[0,1], \|\cdot\|_1)$. *2+2*

(b) Let M be a proper closed subspace of a normed linear space X, $x_0 \notin M$ and $d = d(x_0, M)$. Prove that there is a bounded linear functional f_0 on X such that $\|f_0\| = \frac{1}{d}$, $f_0(x_0) = 1$ and $f_0(M) = 0$. *3*

(c) Show that the right shift operator S on l^2 has no eigenvalue. *3*

4. (a) If X is an infinite dimensional space, show that the set $\{x \in x \mid \|x\| = 1\}$ is not compact. 3

(b) Let X be a Banach space. Show that X is reflexive, if and only if X′ is reflexive. 4

(c) For an orthonormal sequence $\{u_n\}$ in a Hilbert space H, prove the equivalence of the conditions

(i) $x \perp u_n$ for all n implies $x = 0$,

(ii) $x = \sum \langle x, u_n \rangle u_n$ for all $x \in H$. 3

5. (a) Prove that the dual of l' is isometric to l^∞. 4

(b) If f is a bounded linear functional on a Hilbert space H, show that there is a unique $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$. 4

(c) Give an example of a positive operator on l^2. Justify your example. 2

6. Are the following statements *True* or *False* ? Justify your answers with the help of a short proof or a counter example. *5×2=10*

(a) If M is a closed subspace of an inner product space X, then $M^{\perp\perp} = M$.

(b) If D is a dense subspace of a normed linear space X, then $D' \simeq X'$.

(c) Every linear operator $A : \mathbf{R}^n \to \mathbf{R}^n$ is a compact operator.

(d) If a Banach space X is separable, then X′ is separable.

(e) If λ is an eigenvalue of a unitary operator, then $|\lambda| = 1$.