No. of Printed Pages : 4 **MMT-004**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

00691

M.Sc. (MACS)

Term-End Examination

December, 2017

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no. 1 is **compulsory**. Attempt any **four** questions from questions no. 2 to 7. Calculators are **not** allowed. Notations as in the study materials.*

1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) An arbitrary union of closed sets in a metric space is closed.

(b) Any closed and bounded set in a metric space is compact.

(c) The interval (– 1, 1) is nowhere dense in **R**.

(d) The function

$$f(x, y, z, w) = (x^2 - y^2, 2xy, zx, z^2w^2x^2)$$

is a continuously differentiable function on $\mathbf{R}^4$.

(e) $\lim\limits_{n\to\infty} \int f_n \, dm = \int \lim\limits_{n\to\infty} f_n \, dm$ for any sequence of measurable functions $\{f_n\}$.

2. (a) Prove that the coordinate projection maps from $\mathbf{R}^2 \to \mathbf{R}$ are continuous, under the standard metrics on $\mathbf{R}$, $\mathbf{R}^2$. 3

(b) Find the partial derivatives and the total derivative of the function 3

$f(x, y, z, w) = (xy^2, xyz, x^2 + y^2 + zw^2)$ at $(1, 2, -1, 2)$.

(c) For a sequence of non-negative measurable functions f_n, show that 4

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm .$$

3. (a) Let X be a connected metric space and $f : X \to \{0, 1\}$ be a continuous map with respect to discrete topology on $\{0, 1\}$. Show that, either $f(x) = 1$ for all $x \in X$ or $f(x) = 0$ for all $x \in X$. 3

(b) Show by an example that the vanishing of a Jacobian at a point is not a necessary condition for the function to be invertible at that point. 3

(c) Let $A \subseteq \mathbf{R}$ be such that $m^* A = 0$. Prove that

(i) A is measurable

(ii) $m^*(A \cup B) = m^* B \ \forall \ B \subseteq \mathbf{R}$. *4*

4. (a) Define a totally bounded set in a metric space. Show that if X is a totally bounded metric space in which every Cauchy sequence converges, then X is compact. *4*

(b) Obtain the Taylor's series expansion (up to second order) of the function $f(x, y) = x + 2y + xy - x^2 - y^2$ at the point $\left(\frac{4}{3}, \frac{5}{3}\right)$. *4*

(c) Find the components of Q under the standard metric. *2*

5. (a) Let X, Y be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous at a point x_0 if and only if for every sequence $\{x_n\}$ in X converging to x_0, $f(x_n)$ converges to $f(x_0)$ in Y. *5*

(b) Use the Lagrange multiplier method to find and classify the extreme values of the function

$$f(x, y) = 4x + 6y - 2x^2 - 2xy - 2y^2$$

subject to the constraint $x + 2y = 2$, $x, y \geq 0$. *5*

6. (a) Define the compactness of a metric space. Prove that a finite union of compact sets in a metric space is compact. What about arbitrary union ? Justify. *4*

(b) Define Critical points, Stationary points and Saddle points of a function $f : \mathbf{R}^n \to \mathbf{R}$. Find the critical points of the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by $f(x, y) = 2x^4 - 3x^2y + y^2$. Check whether the critical points are saddle points. *3*

(c) Define Time Invariant and Invariant Systems. Give an example for each. *3*

7. (a) Prove that $\mathbf{R}^n$, with the usual metric is a complete metric space. *4*

(b) Can the surface whose equation is $x + y + z - \sin(xyz) = 0$ be described by an equation of the form $z = f(x, y)$ in a neighbourhood of the point (0, 0), satisfying $f(0, 0) = 0$? Justify your answer. *3*

(c) Define a Stable System. Give an example of (i) Stable system (ii) Unstable system. *3*