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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00015

December, 2016

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours

Maximum Marks : 50

(Weightage : 70%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *of the remaining questions. Use of calculator is* ***not*** *allowed.*

1. Are the following statements *True* or *False* ? Justify your answers with a short proof or an example. *5×2=10*

   (a) If linear transformation between normed spaces is continuous, then it is bounded.

   (b) $(C_{00}, \|\cdot\|_1)$ is a Banach space.

   (c) The space $L^1[0, 2\pi]$ is not reflexive.

   (d) The operator $A : C^3 \to C^3$ defined as $A(z_1, z_2, z_3) = (2z_1 + iz_2, 3z_2 + iz_3, -2z_3)$ is not self-adjoint.

   (e) Any two separable Hilbert spaces are linearly isometric.

2. (a) Let $X = C[0, 1]$.

Define $T : X \to X$ by

$Tf(x) = xf(x)$

Show that

(i) T is bounded

(ii) Find $\| T \|$ 3

(b) For $1 \leq i \leq n$, let $(X_i, \| \cdot \|_i)$ be Banach spaces. Let $X = X_T \times X_1 \times X_2 \times \dots \times X_n$ endowed with the norm

$$\| x \| = \sum_{i=1}^{n} \| x_i \|_i .$$

Let $Y = X_1' \times X_2' \dots \times X_n'$ endowed with the norm $\| f \| = \max_{1 \leq i \leq n} \| f_i \|_i$. Show that the dual of X is linearly isometric to Y. 5

(c) Let X be a Banach space and let Y be a closed subspace of X. Let $\pi : X \to X / Y$ be the Canonical quotient map. Show that π is open. 2

3. (a) Define a Schauder basis. Prove that $B = \{e_1, e_2, \dots \}$ is a Schauder basis for C, where C is the space of all convergent scalar sequences with subnorm. Is B a Hamel Basis for C ? Justify your answer. 4

(b) Define the spectral radius of a bounded linear operator $A \in BL(X)$. Find the spectral radius of A in $BL(\mathbf{R}^3)$, where A is given by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, with respect to the standard basis of $\mathbf{R}^3$. *3*

(c) Use Hahn-Banach Theorem to show that a normed linear space X is finite dimensional if its dual X′ is finite dimensional. *3*

4. (a) Let X be an inner product space and $f \in X'$. For any orthonormal set $\{u_\alpha\}$ in X, prove that the set $\{u_\alpha : f(u_\alpha) \neq 0\}$ is a countable set. *3*

(b) Let X be a normed linear space. Let $B(x, r) = \{y \in X : \| x - y \| < r\}$. Show that $\overline{B(x, r)} = \{y \in X : \| x - y \| \leq r\}$. *3*

(c) Prove that a Hilbert Schmidt operator on a Hilbert space is compact. *4*

5. (a) A subspace Y of a Hilbert space H is closed if and only if $Y^{\perp\perp} = Y$. *4*

(b) Let X and Y be normed linear spaces and $F : X \to Y$. If for every Cauchy sequence $\{x_n\}$ in X, the sequence $\{F(x_n)\}$ is Cauchy in Y, then show that F is continuous. 4

(c) Let X be an inner product space over **C** and $x, y \in X$ are such that $\| x + y \| = \| x - y \|$, then show that $x \perp y$. 2

6. (a) Let $T : L^2[0, 2\pi] \to L^2[0, 2\pi]$ be given by

$$Tf(x) = \int_0^{2\pi} \cos(x - t)\, f(t)\, dt.$$

Show that 5

(i) T is self-adjoint.

(ii) cos x and sin x are eigenvectors of T.

(b) Let X be a normed space and E be a subset of X. Show that E is bounded in X, if and only if f(E) is bounded in **K** for every $f \in X'$. 5