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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00984 **December, 2016**

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. 1 is **compulsory**. Attempt any **four** questions out of questions no. 2 to 7. Calculators are **not** allowed.*

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1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) $\{0\} \cup \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ is a compact set in $\mathbf{R}$.

(b) $\mathbf{R}^2 \setminus \{(1, a) : a \in \mathbf{R}\}$ has two components.

(c) The function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x, y) = (x^2 - y^2, 2xy)$ is locally invertible at every point of $\mathbf{R}^2$ except at (0, 0).

(d) The function $f : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $f(x_1, x_2, x_3) = \left(x_1 \sin \frac{1}{x_1}, x_2 - x_3\right)$ is continuously differentiable in $\mathbf{R}^3$.

(e) For f, $g \in L^1([a, b])$, $\| f - g \|$ defined as

$$\| f - g \| = \int_a^b (f(x) - g(x))\, dx \quad \text{is a metric}$$

on $L^1([a, b])$.

2. (a) Is an arbitrary union of closed sets closed ? Justify your answer. 2

(b) Find the stationary points of the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by

$f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz$ and check whether they are local extreme points. 5

(c) When is a set said to be measurable ? Show that the intersection of two measurable sets is measurable. 3

3. (a) Let X be a metric space and $f : X \to Y$ be a function such that for every closed set $V \subset Y$, its inverse image $f^{-1}(V)$ is closed in X. Then show that f is continuous. 4

(b) Let E be the open set such that $E = \{x \in \mathbf{R}^2 : \|x\| < 1\}$ in $\mathbf{R}^2$. Prove that the function $f : E \to \mathbf{R}^3$ given by $f(x_1, x_2) = (e^{x_1}, e^{x_2}, x_1 - x_2)$ belongs to $C^1(E)$. *4*

(c) State the Cantor's intersection theorem. Show that the completeness condition in the theorem cannot be dropped. *2*

4. (a) Check the measurability and integrability of the following functions defined on **R**. Justify your answer. *6*

(i) $f(x) = 2, \quad x = 1, 2, 3, 4$

$= -1, \quad x = -1, -2, -3$

$= 0, \quad$ elsewhere.

(ii) $f(x) = x + e^x$

(iii) $f(x) = \frac{5}{2}, x \in [0, 6]$

$= 0$ elsewhere.

(b) Suppose X and Y are metric spaces with X compact. Prove that a continuous function $f : X \to Y$ is uniformly continuous. *4*

5. (a) Prove that the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ defined by $f(x, y, z, w) = (x + 2y, x^2 - y^2, wz, y + w)$ is locally invertible at (1, 1, 1, 1). *3*

(b) Prove that the Fourier transform of a function in $L^1(\mathbf{R})$ is continuous. *3*

(c) For f, g $\in L^1(\mathbf{R})$, define convolution f * g. Prove that "convolution" is commutative. *4*

6. (a) Suppose X is a compact metric space and Y is any metric space. Suppose $f : X \to Y$ is continuous. Prove that f(X) is compact. *3*

(b) Suppose $f : \mathbf{R}^3 \to \mathbf{R}$ is defined as

$$f(x, y_1, y_2) = x^2 y_1 - e^x + y_2$$

Prove that f satisfies all the conditions of the Implicit Function Theorem near (1, 1). What conclusion can you draw ? *3*

(c) State Monotone Convergence Theorem and show that it is not true for decreasing sequence of functions. *4*

7. (a) (i) Define a linear system.

(ii) Let h be a scalar valued function.

Let $\mathcal{R} : S \to S$ be the system given by

$$(\mathcal{R}f)(t) = \int_{-\infty}^{\infty} h(\tau) f(t - \tau)\, d\tau .$$

Prove that the system R is a linear system, where 'S' is a set of signals. *3*

(b) Suppose $f : E \to \mathbf{R}^n$ is a function which is differentiable at $x \in E$. Suppose the second derivative of f exists at $a \in E$. Prove that the second partial derivatives of all components f_k exist at $a \in E$ for $k = 1, \dots m$ and they satisfy

$$(f_k^2)(a)(e_i, e_j) = \frac{\partial^2 f_k(a)}{\partial x_i \, \partial x_j} = \frac{\partial^2 f_k(a)}{\partial x_j \, \partial x_i}.$$ 4

(c) Find the interior, closure and boundary of the set $A = \{(0, y) \in \mathbf{R}^2 : 0 \le y \le 1\}$ as a subset of $\mathbf{R}^2$ with standard metric. 3