No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00732 **Term-End Examination**

December, 2014

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no.* ***8*** *is* ***compulsory****. Answer any* ***six*** *questions from questions no.* ***1*** *to* ***7****. Use of calculators is* ***not*** *allowed.*

---

1. (a) Customers in a bank may get service from any one of two counters. Customers arrive in the bank according to Poisson law at the rate of 15 per hour. Service time at each counter follows exponential distribution with mean 4 minutes. As an alternative, bank thinks to install an automatic servicing machine which has single service counter but it can serve twice faster than a counter clerk. Obtain average waiting time a customer will have to spend in the bank under both the systems and suggest which system will be better. *7*

(b) Two random variables X and Y have their joint p.d.f., f(x, y) as given below :

$$f(x, y) = \begin{cases} k(x, y) & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(i) the value of k.

(ii) marginal p.d.f. of X and Y.

(iii) and test independence of X and Y. *3*

(c) Let X_1, X_2, X_3 have means 2, 3, 5 and variances 1, 1·5, 1 respectively. If the correlation coefficients $r_{12} = 0{\cdot}5$, $r_{13} = 0{\cdot}4$, $r_{23} = -\,0{\cdot}7$ then write down mean vector and variance-covariance matrix. *5*

2. (a) The transition matrix P of a Markov chain is given below. Obtain P^n and its limiting value for large n.

$$P = \begin{bmatrix} 0{\cdot}2 & 0{\cdot}8 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$$ *5*

(b) Let $X \sim N_4(\mu, \Sigma)$

$$\boldsymbol{\mu} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & 3 & 0 \\ 0 & 9 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \text{ where } \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

(i) Obtain marginal distribution of $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$.

(ii) Test independence of $\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ and $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$.

(iii) Find coefficient of correlation between X_2 and X_4. *5*

(c) A Markov chain has an initial distribution $\mathbf{u}^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \end{bmatrix}$$

(i) Is the chain irreducible ? Give reasons.

(ii) Obtain its stationary distribution.

(iii) Is the stationary distribution unique ? Give reasons. *5*

3. (a) From the past experience the population mean vector and variance-covariance matrix of (X_1, X_2) are as given below :

$$\boldsymbol{\mu} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}, \; \boldsymbol{\Sigma} = \begin{pmatrix} 10 & 5 \\ 5 & 4 \end{pmatrix}.$$

From a sample of size 10, the sample mean vector was found as

$$\bar{\mathbf{X}} = \begin{bmatrix} 22 \\ 6 \end{bmatrix}$$

Test whether the given population mean is a true representation from the sample information at 5% level of significance. (You may use the values $\chi^2_{2,\, 0\cdot05} = 5\cdot99$, $\chi^2_{3,\, 0\cdot05} = 7\cdot81$) 6

(b) In a Branching process the offspring distribution follows binomial law as

$$p_k = \binom{n}{k} p^k q^{n-k};$$

$$k = 0, 1, 2, \ldots, n, \; 0 < p < 1, \; p + q = 1.$$

What is the probability of ultimate extinction of the process given that

(i) $n = 2, \; p = 0\cdot3$

(ii) $n = 2, \; p = 0\cdot6$? 6

(c) Find the matrix of the following quadratic form :

$$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3.$$

And hence identity definiteness of the quadratic form. 3

4\. (a) Suppose $n_1 = 10$ and $n_2 = 15$ observations are taken on variables X_1 and X_2 from two bivariate populations π_1 and π_2. The mean vectors and variance-covariance matrix are given by

$$\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; \boldsymbol{\mu}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{7}{20} \end{bmatrix}.$$

A unit has observation [0, 1]. Classify it to π_1 or π_2 when

(i) costs are equal and prior probabilities are equal.

(ii) prior probabilities are 0·6 and 0·4 and C(1/2) = C(2/1).

(Given is log (2/3) = – 0·4055). 8

(b) There are three counters at the border of a country to check the passports and other documents. If the arrivals at the border be Poisson with λ per hour and service time of each counter be $\lambda/2$ per hour then (i) what is the probability that all the counters are idle ? (ii) what is the expected length of queue ? 3

(c) Obtain the renewal equation when the interarrival distribution is uniform on (0, 1). Also, solve the equation for $t < 1$. 4

5. (a) Let $\{X_n, n = 1, 2, ...\}$ be i.i.d. geometric random variables with probability mass function

$$P(X_n = i) = (1 - p)\, p^{i-1}, \quad i = 1, 2, 3, ...$$

Find the renewal function of the corresponding renewal process. 7

(b) Suppose the random variables X_1, X_2 and X_3 have the covariance matrix

$$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find first, second and third principal components. 8

6. (a) Let the data matrix for a random sample of size $n = 3$ from a bivariate normal population be

$$X = \begin{bmatrix} 6 & 10 & 8 \\ 9 & 6 & 3 \end{bmatrix}.$$

Evaluate the observed T^2 for $\mu_0' = [9, 5]$. What is the sampling distribution of T^2 ? 9

(b) Consider the process $\{X(t), t \in T\}$ whose probability distribution under a certain condition is given by

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n-1}}, \quad n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, n = 0.$$

Find $E(X(t))$ and $var(X(t))$. 6

7. (a) Show that if $\{N_i(t), t \geq 0\}$ are independent Poisson processes with rates λ_i, $i = 1, 2$, then $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$, where $N(t) = N_1(t) + N_2(t)$. 8

(b) If $\mathbf{y}$ be $N_3(\mu, \Sigma)$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

find

(i) distribution of $\mathbf{z} = 2y_1 - y_2 + y_3$

(ii) distribution of $\mathbf{z} = \begin{bmatrix} y_1 + y_2 - y_3 \\ 2y_1 - y_2 + y_3 \end{bmatrix}$

(iii) $r_{12}, r_{13}, r_{23}, r_{23.1}$ 7

8. State whether the following statements are *true* or *false*. Justify your answers with valid reasons. *10*

(a) The relation of accessibility in states is transitive.

(b) Let X_1 and X_2 be bivariate normal variables and are uncorrelated, then X_1 and X_2 are not necessarily independent.

(c) If $A = \begin{pmatrix} 3 & -1 \\ -1 & -2 \end{pmatrix}$, then A is positive semi-definite.

(d) In a renewal process the durations between successive occurrences of events are not necessarily independent.

(e) In a Markov chain a state will be persistent if in a long run the return to the state is certain.