No. of Printed Pages : 4 **MMT-006**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00072

December, 2014

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. **1** is **compulsory**. Attempt any **four** of the remaining questions. Use of calculators is **not** allowed.*

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1. Are the following statements *true* or *false* ? Justify your answer with the help of a short proof or counter example. *5×2=10*

(a) l^3 is not a Hilbert space.

(b) A compact operator is never invertible.

(c) l^∞ is not separable.

(d) C^1 [0, 1] is complete with sup norm.

(e) Every non-zero bounded linear map is open.

2. (a) If A is a bounded self-adjoint operator on a Hilbert space H, show that $A + iI$ is invertible. 3

(b) Let $(X_1, \| \cdot \|_1)$, $(X_2, \| \cdot \|_2)$ be Banach spaces. On $X_1 \times X_2$ define

$$\| (x_1, x_2) \| = \| x_1 \|_1 + \| x_2 \|_2.$$

Prove that this is a norm and $X_1 \times X_2$ is complete with this norm. 4

(c) Let X be a normed linear space. Suppose $x_n \to x$ in X and (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$. Show that $\lambda_n x_n \to \lambda x$ in X. 3

3. (a) State the closed graph theorem. Prove that a bounded linear map has a closed graph. Is the converse always true ? Justify. 5

(b) Consider the linear space $X = \mathbf{R}^2$ with the norm $\| \cdot \|_1$ given by

$$\| x \|_1 = | x_1 | + | x_2 |, \ x = (x_1, x_2) \in \mathbf{R}^2.$$

Let G denote the subspace of $\mathbf{R}^2$ given by $G = \{(x_1, 0) : x_1 \in \mathbf{R}\}$ and f be the linear functional defined on G by

$$f(x_1, 0) = \alpha x_1 \text{ where } \alpha > 0.$$

Show that $\tilde{f} : x \to \mathbf{R}$ defined by

$$\tilde{f}(x) = \alpha x_1 + \frac{\alpha}{2} x_2, \ x = (x_1, x_2) \in X$$

is a Hahn-Banach extension of f to X. Find another Hahn-Banach extension. 5

4. (a) Prove that the norm limit of a sequence of compact operators is compact. *4*

(b) Let M and N be closed linear subspaces of a Hilbert space H. Determine $(M \cap N)^{\perp}$ in terms of $M^{\perp}$ and $N^{\perp}$. *3*

(c) Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a linear space X. If $(X, \|\cdot\|_1)$ is complete, show that $(X, \|\cdot\|_2)$ is also complete. *3*

5. (a) Prove that a closed linear subspace of a reflexive space is reflexive. *3*

(b) Give an example of a compact self-adjoint operator on l^2. Find an eigenvalue for this example. *4*

(c) Calculate the norm of the linear functional $f : (\mathbf{R}^2, \|\cdot\|_2) \to \mathbf{R}$, $f(x_1, x_2) = x_1 - x_2$. *3*

6. (a) Let M be a closed linear subspace of a Banach space X. Prove that the quotient space X/M, with the usual norm, is complete. *3*

(b) Consider the space C[–1, 1] of real valued continuous functions of [–1, 1] with the inner product

$$< x, y > = \int_{-1}^{1} x(t)\, y(t)\, dt, \ (x, y) \in [-1, 1].$$

If M is a subspace of even functions in C[–1, 1], find $M^{\perp}$. *3*

(c) Give an example of a bounded linear operator on l^2 with no eigenvalue. *4*

7. (a) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that for a fixed $y \in l^q$, there exists an $f_y \in l^{p'}$ such that $\| f_y \| = \| y \|_q$. *4*

(b) Consider $\mathbf{R}^2$ with $\| \cdot \|_2$. Let $M = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\}$ and $x = (1, -1)$. Find d(x, M). *2*

(c) Prove that a bounded linear operator A on a Hilbert space H is normal if and only if $\| Ax \| = \| A^*x \|$ for all x in H. *4*