No. of Printed Pages : 6 **MMT-008**

# M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

**Term-End Examination** 00641

**December, 2012**

## MMT-008 : PROBABILITY AND STATISTICS

*Time : 3 hours* *Maximum Marks : 100*

*(Weightage : 50%)*

***Note :*** *Question No. 8 is* ***compulsory****. Answer* ***any six*** *questions from question No. 1 to 7. Use of calculator is* ***not*** *allowed.*

**1.** (a) Random variables X and Y have joint p.d.f 7

$$f(x, y) = \frac{1}{8}\,(y^2 - x^2)\,e^{-y}\,;\ |X| \le y,\ y > 0.$$

(i) Find marginal distributions of X and Y and hence prove that X and Y are dependent.

(ii) Check whether X and Y are correlated or not.

(b) Using the method of spectral decomposition evaluate $p^n$ for the following transition matrix 8

$$P = \begin{bmatrix} 0.7 & 0.3 \\ .6 & 0.4 \end{bmatrix}.$$

Hence obtain $\lim\limits_{n \to \infty} p^n$.

2. (a) Differentiate between irreducible and a periodic Markov Chain with the help of examples. Define recurrence and transient states. If n step transition probabilities of a given state i is $p_{ij}^{(n)} = \frac{1}{2}$ for n > No then 6

   (i) Show that all the states are a periodic and non-null persistent.

   (ii) Find the mean recurrence time for the state i.

(b) A system has two identical components, but uses only one at a time. If first one fails, it automatically switches on to the second. The system fails, when both the components fail. At that time both are replaced instantaneously. Assuming that the life time distribution of the component is exponential with parameter $\lambda$, express the system in a renewal process frame work. Find the inter-occurance time distribution and its laplace transform. 9

3. (a) Consider a planned replacement policy that takes place every 3 years ie. a machine required for continuous use is replaced on failure or at the end of three years. Compute. 7

   (i) long term rate of replacement

   (ii) long term rate of failure

   (iii) long term rate of planned replacement.

   Assume that successive machine life times are uniformly distributed over the interval [2, 5] years.

(b) Describe M/M/K/N queueing system, stating the assumptions. Derive the expression for probability of i customers in the system in steady state. Also, derive expression for expected number of customers in the queue. 8

4. (a) Write the assumptions for the system M/D/2/N in queueing notation. In a single server channel in doctor's clinic, patients arrive and join the queue at the end every $\alpha$ minutes and leave the clinic every $\beta$ minutes. Assuming the queue with n patients, find the queue length ? 5

(b) Consider the mean vector $\mu_x = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and 10

$\mu_y = 9$, and the covariance matrices of $x_1$, $x_2$ and $y$ are

$$\sum_{xx} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \ \sigma_{yy} = 10, \ \sigma_{xy} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(i) Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation.

(ii) Find the multiple correlation coefficient

(iii) Find the mean square error.

5. (a) If A is a real symmetric matrix of order n, prove that there exists a real orthogonal matrix P of order n such that $A = PDP'$ where D is a real diagonal matrix. Further show that elements of D are eigen values of A and the columns of P are orthonormal eigen vectors of A. 7

(b) In answering a question on a multiple choice test an examinee either knows the answer (with probability p) or guesses (with probability (1-p)). Assume that the probability of answering a question correctly is unity if examinee knows the answer and 1/m if the examinee guesses. Suppose an examinee answers a question correctly, What is the probability that the examinee really know the answer ? 5

(c) Let $X \sim N_3(\mu, \Sigma)$, where $\mu_x = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and 3

$\Sigma = \begin{bmatrix} 9 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}$. Obtain the conditional

distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given $x_3 = 1$.

6. (a) $X = (X_1, X_2, X_3)'$ has $N_P(\mu, \Sigma)$ distribution, 8

where $\mu = (1, -1, 0)'$ and $\Sigma = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

Find

(i) conditional distribution of $X_1$ given $X_2 = -0.5$ and $X_3 = 0.2$.

(ii) Obtain a non singular transformation $Y = TX + C$ such that $Y_1$, $Y_2$ and $Y_3$ are independent standard normal variables.

(b) Find principle components and proportion of total population variance explained by each when the covariance matrix is : 7

$$\Sigma = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

7. (a) Consider three random variables $X_1$, $X_2$ and $X_3$ having positive definite covariance matrix given by : 7

$$\Sigma = \begin{bmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{bmatrix}.$$

Write its factor model with single factor and show that no proper solution exists in this case.

(b) Determine the definiteness of the following quadratic forms (give details) : 4

(i) $x_1^2 - x_2^2 + x_3^2 - x_1 \ x_2 + 10 \ x_1 \ x_3 - 2x_2 \ x_3$

(ii) $9 \ x_1^2 + 4x_2^2 + 4x_3^2 + x_4^2 + 8 \ x_2 \ x_3 + 12 \ x_3 \ x_1 + 12 \ x_1 \ x_2$

(c) Let the data matrix for a random sample of size $n = 3$ from a bivariate normal 4

population be $X = \begin{bmatrix} 6 & 10 & 8 \\ 9 & 6 & 3 \end{bmatrix}$.

Evaluate the observed $T^2$ for $\mu_0' = [9, 5]$. What is the sampling distribution of $T^2$ in this case ?

8. State whether the following statements are true or false. Justify your answers : **2x5=10**

(a) Irreducible recurrent Markov Chain always has stationary distributions.

(b) The processes $\{Nt \quad t \geqslant 0\}$ and $\left\{N_{t+x_1} - 1, t \geqslant 0\right\}$ have the same distribution.

(c) In the queueing system represented by $M/E_5/3/15/20/FCFS$ the distribution of service time is exponential.

(d) The unique square root of a matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \text{ is } \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{7}}{2} \\ -\frac{1}{2} & \frac{\sqrt{7}}{2} \end{bmatrix}.$$

(e) Let $X_i \sim N_p(\mu, \Sigma)$, where $i = 1, 2, 3 \_\_\_ N$. be independent, then $\overline{X} \sim N_p(\mu, \Sigma)$.