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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

01903

Term-End Examination

December, 2012

MMT-004 : REAL ANALYSIS

Time : 2 hours | *Maximum Marks : 50*

Weightage : 70%

*Note : Question No. **1** is **compulsory**. Do **any four** questions out of questions **2** to **7**. Calculators are not allowed.*

1. State, giving reasons, whether the following statements are *true* or *false*. **5x2=10**

   (a) If (X_1, d_1) and (X_2, d_2) are two discrete metric spaces, then the product metric space $X_1 \times X_2$ is discrete.

   (b) The set $E := \bigcup_{n=1}^{\infty} \{x \mid \sin n\pi x = 0\} \cap [0, 1]$

   is Lebesgue measurable.

   (c) The function $f : \mathbf{R} \to \mathbf{R}^2$ defined by $f(x) = (f_1(x), f_2(x))$

   $\underline{x \in R}$

   where $f_1(x) = x$ and $f_2(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

   is differentiable of O.

(d) $\int_0^1 \left(\lim_{n\to\infty} f_n(x) \right) dx = \lim_{n\to\infty} \int_0^1 f_n(x)\, dx$

where $f_n(x) = n.\, \chi_{\left(0, \frac{1}{n}\right)}^{(x)}$.

(e) The set $E := \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbf{N} \right\} \subset \mathbf{R}$ is not compact.

2. (a) Let E_1 and E_2 be open sets in a metric space X. Show that $E_1 \cap E_2$ is open in X. Is $\bigcap_{1=1}^{\infty}$ Ei open in X^- if each E_i is open in X ? Justify your answer. 3

(b) If $f(x, y, z, w) = (x^2 - y^2, 2xy, zx, z^2w^2x^2)$ and $v = (2, 1, -2, -1)$, find $D_v f(1, 2, -1, -2)$. 3

(c) Let $A \subseteq \mathbf{R}$ be such that $m^*A = 0$. Prove that 4

(i) A is measurable

(ii) $m^*(A \cup B) = m^*B \quad \forall\, B \subseteq \mathbf{R}$

3. (a) State dominated convergence theorem. Use the dominated convergence theorem to find 4

$$\lim_{n\to\infty} \int_0^1 \frac{n\sqrt{x}}{1 + n^2 x^2}\, dx$$

(b) Show that the function f defined from $\mathbf{R}^4 \to \mathbf{R}^4$ by $f(x, y, z, w) = (2x - y, x^2 + yz, xz + 3w, x + w^2)$ is locally invertible at $(0, 1, 2, -1)$. 3

(c) State Urysohn's Lemma. Use it to show that if A and B are disjoint closed subsets of a metric space (X, d) then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. 3

4. (a) Prove that in a discrete metric space a set is compact if and only if it is finite. 3

(b) Consider the system $R : f \to g$ given by 4

$$g(t) = (Rf)(t) = \int_{-\infty}^{t} f(\tau) e^{-(t-\tau)} d\tau$$

(i) Is this a memory less system ?

(ii) Is it stable system ? Justify your answers.

(c) Find the Fourier series of the function f defined by 3

$$f(x) = \begin{cases} -x^2, & -\pi < x \leq 0 \\ x^2, & 0 < x < \pi \end{cases}$$

5. (a) State Implicit Function theorem for $\mathbf{R}^3$. 5
Consider the function $f : \mathbf{R}^3 \to \mathbf{R}$ defined by
$f(x, y_1, y_2) = x^2 y_1 - e^x - y_2$.
Show that there exists a differentiable function g in some neighbourhood of $(1, -1)$ in $\mathbf{R}^2$
such that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$.

(b) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbf{R}$. Does it converge in $\mathbf{R}$? Justify your answer. 3

(c) Which of the following sets are nowhere dense ? 2

(i) $[-1, 1]$

(ii) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\ldots\right\}$

6. (a) Prove that the continuous image of a connected metric space is connected. 3

(b) (i) Show that every totally bounded metric space is bounded.

(ii) Let X be an infinite set with discrete metric d. Show that (X, d) is a bounded metric space but is not totally bounded. 4

(c) Show that all real valued continuous functions defined on [0, 1] are bounded. Is this result true for real valued continuous functions defined on (0, 1] ? Justify your answer. 3

7. (a) Find the extreme values of the function **4**

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

subject to the constraint

$$4x_1 + x_2^2 + 2x_3 = 14,$$

$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,$$

(b) Give an example to show that arbitrary union of compact sets need not be compact. **3**

(c) If a set E has finite measure (i.e. $m(E) < \infty$) then, show that, $L^2(E) \subset L^1(E)$. **3**
