M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

December, 2012

MMT-003 : (ALGEBRA)

Time : 2 hours   *Maximum Marks : 50*

Weightage 70%

*Note :   Question no. 1 is **compulsory**. Answer **any four** questions from q. 2 to q. 6.*

1.   State with reasons, which of the following   **10**
     statements are **true** and which are **false**.

   (a)   A nontrivial group homomorphism
         $\varphi : A_5 \to G$ is injective.

   (b)   The group $Z_5 \times Z_{10} \times Z_3$ is isomorphic to
         $Z_{25} \times Z_{15}$.

   (c) Z_{25} is not a field.

   (d) $10 = 1 + 2 + 3 + 4$ is the class equation of
         some group of order 10.

   (e)   A group of order 25 cannot have a
         representation of degree 3.

2. (a) Prove that the 2-Sylow subgroup of D_6 is **4** not cyclic.

(b) For a finite field F of 17 elements, show that **3** F [$\sqrt[3]{10}$] and F [$\sqrt[3]{9}$] are not the same.

(c) Let $G = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbf{Z} \right\}$ and $S = \mathbf{Z}^2$. **3**

Check that $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + b\,m \\ b \end{pmatrix}$

defines a group action of G on S. Describe

the orbit of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ under this action.

3. (a) Show that the polynomial $x^3 - x + 2$ over the **4** finite field $\mathbf{F}_3$ is irreducible check that, if α is any root of $x^3 - x + 2$, $\alpha - 2$ is also a root of $x^3 - x + 2$.

(b) Prove that the symmetric group S_8 has no. **3** element of order 14.

(c) Suppose that a given group of order 72 has **3** an irreducible representation of degree 8, then show that it should have a nontrivial irreducible representation of degree ≤ 2.

4. (a) Solve the following simultaneous   5
congruences.

$$5\,x \equiv 7 \bmod 9$$
$$x \equiv 2 \bmod 5$$
$$x \equiv 3 \bmod 4$$

(b) Let $D_4 = \{\, x,\, y \mid x^4 = 1,\; y^2 = 1 \mid yx\,y = x^3 \,\}$ 5
check that $\rho : D_4 \to GL_2\,(\mathbf{C})$ defined by

$$\rho\,(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \rho\,(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ is a}$$

representation of D_4. Find the character of
ρ. Check whether ρ is irreducible.

5. (a) Is $Q\left(\sqrt{6}\right) \subseteq Q\left(\sqrt{2} + \sqrt{3}\right)$? Justify your   3
answer.

(b) Prove that the following matrices are   5
symplectic :

(i) $\begin{bmatrix} & -I \\ I & \end{bmatrix}$ where I is the $n \times n$ identity
matrix.

(ii) $\begin{bmatrix} A^t & \\ & A^{-1} \end{bmatrix}$ where A is a $n \times n$

invertible matrix.

(c) For two elements a, b in a group h, show   2
that ab and ba have the same order.

6. (a) Verify that $978 - 981 - 4304 - 96 - 2$ is a valid ISBN/number. **3**

 (b) Find the minimal polynomial satisfied by $\sqrt{2} + \sqrt{5}$. over Q. **5**

 (c) Define a regular language. Give an example of a regular language. **2**

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