M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

00683

M.Sc. (MACS)

Term-End Examination

December, 2013

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*
Weightage 70%

Note : *Answer question **number 1** which is compulsory. Attempt any four from the remaining six.*

1. Are the following statements **true or false** ? Justify your answer with the help of a short proof or a counter example. **5x2=10**

   (a) If A is a normal operator and B is an unitary operator on a Hilbert space H, then AB is a normal operator on H.

   (b) A normal space X is separable implies that the dual space X' is separable.

   (c) Every absolutely convergent series in a Banach space is convergent.

   (d) If A and B are non-empty subsets of an inner product space X and $B \subset A$, then $B^{\perp} \subset A^{\perp}$.

   (e) Every linear map on a normed space is bounded.

2. (a) Let X be a normed space and $a \in X$. Show that there exists an $f \in X'$ such that $\|a\| = \text{Sup}\{ |f(a)| : f \in X', \|f\| \leq 1 \}$. **3**

(b) Let X be a normed space and Y be a Banach space and $X \neq \{ 0 \}$. Show that $B<(X, Y)$ is a Banach space. 4

(c) Let A be a bounded linear operator on a Hilbert space H and $\|A(x)\| = \|x\| \ \forall \ x \in H$. Show that A is unitary. 3

3. (a) Define a reflexive normed space. Show that l^p is reflexive for $1 < p < \infty$. Is l' reflexive ? Justify your answer. 6

(b) When are two norms on a linear space said to be equivalent ? Give two inequivalent norms on $C'[0, 1]$. Justify your answer. 4

4. (a) Let X be a finite dimensional normed space and $A : X \to X$ is a linear map. Show that A is compact. Is this result true if X is infinite dimensional and A is a continuous linear map ? Justify. 5

(b) Let H be a Hilbert space and $A \in B<(H)$ be unitary. If $\{u_\alpha\}$ is an orthonormal set in H, show that $\{ A(u_\alpha) \}$ and $\{ A^*(u_\alpha) \}$ are orthonormal sets. 3

(c) Let X, Y be normed spaces and $T : X \to Y$ be a bounded linear map. Show that T is uniformly continuous. 2

5. (a) Show that the spectrum of a bounded linear map may be empty. Also give an example to show that there are operators A, with $\sigma(A) \neq \phi$ but $\sigma_e(A) = \phi$. 5

(b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Define the transpose F' of F. If F is continuous show that F' is continuous and $\|F'\| = \|F\|$. 3

(c) Let $X = \mathbb{R}^n$. 2
Let $\|x\|_\infty = \max\{|\alpha_1|, |\alpha_2| \ldots\ldots, |\alpha_n|\}$, $x = (\alpha_1, \alpha_2 \ldots\ldots \alpha_n) \in \mathbb{R}^n$.
Show that $\|\cdot\|_\infty$ is a norm on X.

6 (a) Let X and Y be normed spaces and $F : X \to Y$ be a continuous linear map. Show that F is closed . Is the converse true ? Justify. 4

(b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is continuous if and only if every cauchy sequence $\{x_n\}$ in X, the sequence $\{F(x_n)\}$ is cauchy in Y. Show that this is not true for non-linear continuous map. 6

7. (a) Let F be a finite dimensional subspace of an inner product space X. Then show that $X = F + F^\perp$ and $F^{\perp\perp} = F$ 3

(b) Let $X = L^2[0, 1]$ and $\phi \in L^\infty[0, 1]$ and A be the operator on X given by $Ax = \phi x$, $x \in X$ show that the operator $B : X \to X$ defined by $Bx = \overline{\phi}\, x$, $x \in X$, is the adjoint of A. Where $\overline{\phi}$ is the complex conjugate of ϕ. 3

(c) State open mapping theorem. Show by an example that the theorem may not hold if any of the normed spaces are not Banach. 4