MMT-008

ASSIGNMENT BOOKLET

M.Sc. (Mathematics with Applications in Computer Science)
PROBABILITY AND STATISTICS
(Valid from 1st July, 2020 to 30th June, 2021)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
2020 - 21

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 20 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE : …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE : …………………………..….. DATE : ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto June, 2021. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by June, 2021, then you need to get the assignment for 2021-22 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment
(To be done after reading all the blocks)

Course Code: MMT-008
Assignment Code: MMT-008/TMA/2020-21
Maximum Marks: 100

Q 1. (a) Suppose the joint density function of X and Y is given by

$$f(x,y)=\begin{cases}4y(x-y)e^{-(x+y)}, & 0<x<\infty, 0\le y\le x\\ 0, & \text{otherwise}\end{cases}$$

Compute $E\left[X \middle| Y=y\right]$. (5)

(b) Let $X'=[X_1,X_2]$ be a random vector with mean vector $\mu'_X=[\mu_1,\mu_2]$ and variance-covariance matrix

$$\Sigma_x=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$$

Find the means and covariance matrix for the linear combinations

$$Z_1 = X_1 - X_2$$

$$Z_2 = X_1 + X_2$$

or

$$Z=\begin{bmatrix}Z_1\\ Z_2\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}=CX$$

in terms of μ_x and Σ_x. (5)

2. a) Suppose the events of a Poisson-process $\{N(t)\}$ are classified as belonging to category $i\,(i=1,2,\ldots\ldots,k)$ with probabilities p_i independently of $N(t)\left(\sum p_i=1\right)$. Let $N_i(t)$ be the number of events of category i during $(0,t]$. Show that

i) $\{N_i(t)\}$ is a Poisson process with rate λp_i, $(i=1,2,\ldots\ldots,k)$.

ii) $N_1(t),\ldots\ldots, N_k(t)$ are mutually independent.

(*Hint*: For given $N(t)=n, (N_1(t),\ldots\ldots, N_k(t))$ are multinomially distributed.) (6)

b) A machine component is found to have life-time distribution with probability density function

$$f(x)=\frac{x}{100}e^{-x/10},(x>0).$$

Show that the renewal function is given by

$$\frac{t}{20}-\frac{1}{2}+\frac{1}{2}e^{-t/10},$$

if the process starts with a new component. (4)

3. The joint probability mass function of X and Y, $p(x, y)$, is given by

$$p(1,1)=\frac{1}{4},\quad p(2,1)=\frac{1}{4},\quad p(3,1)=\frac{1}{16}$$
$$p(1,2)=\frac{1}{16},\quad p(2,2)=0,\quad p(3,2)=\frac{1}{16}$$
$$p(1,3)=0,\quad p(2,3)=\frac{1}{16},\quad p(3,3)=\frac{1}{4}$$

i) Compute $E[X \mid Y=i]$ for $i = 1, 2, 3$.
ii) Find P[x/y=1]
iii) Marginal distributions of X and Y. (10)

4. a) For a series of dependent trials the probability of success on any trial is $(k+1)/(k+2)$ where k is equal to the number of successes on the previous two trials. Compute $\lim_{n\to\infty} P$ [success on the n th trial]. (5)

b) For a branching process, calculate π_0 when

i) $P_0=\frac{1}{5},\ P_2=\frac{4}{5}$
ii) $P_0=\frac{1}{5},\ P_1=\frac{3}{5},\ P_2=\frac{1}{5}$
iii) $P_0=\frac{1}{6},\ P_1=\frac{1}{2},\ P_3=\frac{1}{3}$. (5)

5. a) Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ. The bus departs at time t. Let X denote the total amount of waiting time of all those that get on the bust at time t. We want to determine $\text{Var}(X)$. Let $N(t)$ denote the number of arrivals by time t.

i) What is $E[X \mid N(t)]$?
ii) Argue that $\text{Var}[X \mid N(t)] = N(t)t^2/12$.
iii) What is $\text{Var}(X)$? (5)

b) Consider a birth and death process with birth rate $\lambda_i=(i+1)\lambda,\ i\ge 0$, and death rates $\mu_i = i\mu,\ i \ge 0$.

i) Determine the expected time to go from state 0 to state 4.
ii) Determine the expected time to go from state 2 to state 5.
iii) Determine the variances in parts (i) and (ii). (5)

6. Twenty-five portfolio managers were evaluated in terms of their performance. Suppose Y represents the rate of return achieved over a period of time; Z_1 is the manager's attitude toward risk measured on a five-point scale from "very conservative" to "very risky"; and Z_2 is years of experience in the investment business. The observed correlation coefficients between pairs of variables are:

$$
\begin{array}{c} \\ \boldsymbol{R} = \end{array}
\begin{array}{c} \quad Y \qquad\quad Z_1 \qquad Z_2 \\ \begin{bmatrix} 1.0 & -.35 & .82 \\ -.35 & 1.0 & -.60 \\ .82 & -.60 & 1.0 \end{bmatrix}. \end{array}
$$

i) Interpret the sample correlation coefficients $r_{YZ_1} = -.35$ and $r_{YZ_2} = .82$.

ii) Calculate the partial correlation coefficient $r_{YZ_1 \cdot z_2}$ and interpret this quantity with respect to the interpretation provided for r_{YZ_1} in Part (i). (10)

7. Consider the two data sets

$$X_1 = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 5 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 5 & 7 \end{bmatrix}$$

for which

$$\bar{x}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$S_{pooled} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

i) Calculate the linear discriminant function.

ii) Classify the observation $x_0' = [2, 7]$ as population π_1 or population π_2 with equal priors and equal costs. (10)

8. A single repairperson looks after the two machines 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate λ_i, where $i = 1, 2$. When machine i fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. The repairperson will always service machine 1 when it is down. For instance, if machine 1 fails while 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start on 1.

(i) Write down all the states.
(ii) What is the probability that the machine 2 is down. (10)

9 a) Let X have covariance matrix

$$\Sigma = \begin{bmatrix} 16 & -2 & 4 \\ -2 & 9 & 1 \\ 4 & 1 & 25 \end{bmatrix}$$

i) Determine ρ and $V^{1/2}$

ii) Multiply your matrices to check the relation $V^{1/2} \rho V^{1/2} = \Sigma$. (4)

b) Suppose the random variables X_1, X_2, and X_3 have the covariance matrix

$$\Sigma = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find all principal components. (6)

8. Which of the following statements are true or false. Give reasons. (10)

a) $Q(x) = x_1^2 - x_2^2$ is the quadratic form of a positive definite matrix.

b) In a bivariate normal distribution $N_2(\mu_x, \mu_y, \sigma_x, \sigma_y \int xy)$, if $\int xy = 0$, then x and y are independent.

c) The relation of accessibility in states is transitive.

d) For any pair of discrete random variables X and $Y \sum_y P[Y / y \mid X / x] < 1$.

e) If $\{X(t) : t \geq 0\}$ is a Poisson process, then $N(t) = X(t + a) - X(t)$, where a is a constant is also a Poisson process.