MTE-06

ASSIGNMENT BOOKLET

Bachelor's Degree Programme

ABSTRACT ALGEBRA

(Valid from 1st January, 2020 to 31st December, 2020)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
Indira Gandhi National Open University
Maidan Garhi
New Delhi-110068
(2020)

Dear Student,

Please read the section on assignments in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment** for this course. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully:

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.: ……………………………………………

NAME: ……………………………………………

ADDRESS: ……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE: …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE: …………………………..….. **DATE:** ……………………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) While solving problems, clearly indicate which part of which question is being solved.

6) **This assignment is valid only upto December, 2020**. If you have failed in this assignment or fail to submit it by the last date, then you need to get the assignment for the next cycle and submit it as per the instructions given in that assignment.

7) It is compulsory to submit the assignment before filling in the exam form.

We strongly suggest that you retain a copy of your answer sheets.

We wish you good luck.

ASSIGNMENT

(To be done after studying all 4 blocks.)

Course Code: MTE-06
Assignment Code: MTE-06/TMA/2020
Maximum Marks: 100

1) Which of the following statements are true? Give reasons for your answers, in the form of a short proof or a counterexample.

i) $\mathbb{M}_3(\mathbb{Z})$ has no nilpotent elements.

ii) If P_1 and P_2 are prime ideals of a ring R, then $P_1 P_2 = P_1 \cap P_2$.

iii) The set of cosets of $\langle (1 \quad 2) \rangle$ in S_3 is a group with respect to multiplication of cosets.

iv) If $(G,*)$ is a group, then $*$ is the only binary operation defined on G.

v) If every element of a group G has finite order, then G must be of finite order.

vi) If k is a field, so is $k \times k$.

vii) x is a unit in $\mathbb{R}[x]$.

viii) If A and B are two sets such that $A \cup B = \emptyset$, then $A \cap B = \emptyset$.

ix) $\mathbb{Q}[x] \big/ \langle x^6 + 17 \rangle$ is a field of characteristic 6.

x) Any two groups of order m are isomorphic, where $m \in \mathbb{N}$. (20)

2) a) Show that if $f : \mathbb{Q} \to \mathbb{Q}$ is a ring homomorphism, then $f(x) = x \ \forall\, x \in \mathbb{Q}$. Would this still be true if f were a group homomorphism? Why, or why not? (6)

b) Prove that $\mathbb{R}^5 \big/ \mathbb{R} \simeq \mathbb{R}^4$ as rings. (7)

c) Give a non-trivial element of the ring $\mathbb{Z}_{10} / < \overline{4} >$, with justification. (2)

3) a) Is $R = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \middle|\, a, b \in \mathbb{Z} \right\}$ a subring of $\mathbb{M}_2(\mathbb{Z})$? Why, or why not? (2)

b) Let R be a ring for which $ab = ca \Rightarrow b = c \ \forall\, a, b, c \in R, a \neq 0$. Show that R is commutative. (2)

c) Let R be a ring. Show that $\mathbb{M}_3(R)$ is a ring with respect to the usual matrix addition and multiplication. Further, if R is commutative, will $\mathbb{M}_3(R)$ be commutative? Why, or why not? (8)

d) Check whether or not $S = \{a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x] \,|\, 5 \,|\, a_0\}$ is an ideal of $\mathbb{Z}[x]$. (3)

4. a) Show that if G is a non-cyclic group of order n, then G has no element of order n. Further, give an example, with justification, of a non-cyclic group with all its proper subgroups being cyclic. (4)

b) Let G be a group and H be a non-empty finite subset of G. If $ab \in H \, \forall \, a, b \in H$, then show that $H \leq G$. Will the result remain true if H is not finite? Give reasons for your answer. (6)

5. a) Obtain the order of each element of $\wp(S)$, where $S = \{1, 2, 3\}$. (4)

b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$ in S_7. Write $\sigma\tau$ as a product of disjoint permutations.
Further, is $\sigma\tau$ even? Why, or why not? (3)

c) Define $\sim$ on $\mathbb{R}$ by '$a \sim b$ iff $a - b \in \mathbb{Z}$'. Check whether or not $\sim$ is an equivalence relation on $\mathbb{R}$. If it is, find $[\sqrt{5}]$. Else, give another equivalence relation on $\mathbb{R}$. (3)

6. a) Find $Z(D_8)$, the centre of D_8. Also give the algebraic structure of $D_8/Z(D_8)$. (5)

b) Find the number of normal subgroups of order 25, and of order 50, of a group of order 75. (3)

c) Check whether or not $\mathbb{R}$ is a normal subgroup of the group $\mathbb{H} = (\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k, +)$, where $i^2 = j^2 = k^2 = -1, ij = -ji, jk = -kj, ki = -ik$ and $(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k$ for $a, b, c, d, a', b', c', d' \in \mathbb{R}$. (2)

7. a) Consider the ideal $I = \left\langle x^3 - 1, 2x^4 + 2x^3 + 7x^2 + 5x + 5 \right\rangle$ in $\mathbb{Q}[x]$. Find $p(x) \in \mathbb{Q}[x]$ such that $I = \langle p(x) \rangle$.
Is $\mathbb{Q}[x]/I$ a field? Give reasons for your answer. (4)

b) Show that $d : \mathbb{Q}[x] \setminus \{0\} \to \mathbb{N} \cup \{0\} : d(f) = 5^{\deg f}$ is a Euclidean valuation on $\mathbb{Q}[x]$. (4)

b) Check whether or not $\left\langle (x+1)^2 \right\rangle$ is a maximal ideal of $\mathbb{Z}[x]$. (2)

8. a) Show that $\mathbb{Z}[\sqrt{-2}]$ is not a UFD. (7)

b) Use the ring in 8(a) above to show that
i) the quotient ring of a UFD need not be a UFD;
ii) an irreducible element of a UFD need not be a prime element. (3)