MMT-006

ASSIGNMENT BOOKLET

(Valid from 1st January, 2025 to 31st December, 2025)

M.Sc. (Mathematics with Applications in Computer Science)

FUNCTIONAL ANALYSIS

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2025)

Dear Student,

Please read the section on assignments in the Programme Guide that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment** for this course. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME:……………………………………………

ADDRESS:……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE: …………………………….

ASSIGNMENT NO. …………………………..…

STUDY CENTRE: ………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) While solving problems, clearly indicate which part of which question is being solved.

6) This assignment is **valid from 1st Jan, 2025 to 31st Dec, 2025**. If you have failed in this assignment or fail to submit it by Dec, 2025, then you need to get the assignment for the year 2026, and submit it as per the instructions given in the Programme Guide.

7) **You cannot fill the examination form for this course** until you have submitted this assignment.

We strongly suggest that you retain a copy of your answer sheets.

We wish you good luck.

Assignment

Course Code: MMT-006
Assignment Code: MMT-006/TMA/2025
Maximum Marks: 100

1. State whether the following statements True or False? Justify your answers: 5×2 =10

   a) The function $\|\cdot\|$ defined on $\mathbb{R}^n$ as:

   $$\| x \| = \sum_{j=1}^{n} |a_j| \text{ for } x = (a, a_2, \ldots., a_n) \in \mathbb{R}^n$$

   is a norm.

   b) C_0 is a Banach space.

   c) If A is the right shift operator on l^2, then the eigen spectrum is non-empty.

   d) If a normed linear space is reflexive, then so is its dual space.

   e) If a normed linear space X is finite dimensional, then so is X'.

2. a) Consider the space c_{00}. For $x = (x_1, x_2, \ldots, x_n, \ldots) \in c_{00}$, define $f(x) = \sum_{n=1}^{\infty} x_n$. Show that f is a linear functional which is not continuous w.r.t the norm $\| x \| = \sup_n |x_n|$. (5)

   b) Consider the space $C^1[0,1]$ of all C^1 functions on $[0,1]$ endowed with the uniform norm induced from the space $C[0,1]$, and consider the differential operator $D : (C^1[0,1], \|.\|_\infty) \rightarrow (C[0,1], \|.\|_\infty)$ defined by $Df = f'$. Prove that D is linear, with closed graph, but not continuous. Can we conclude from here that $C^1[0,1]$ is not a Banach space? Justify your answer. (5)

3. a) When is a normed linear space called separable? Show that a normed linear space is separable if its dual is separable [You should state all the proposition or theorems or corollaries used for proving the theorem]. Is the converse true? Give justification for your answer. [Whenever an example is given, you should justify that the example satisfies the requirements.] (6)

   b) Let X be a Banach space, Y be a normed linear space and $\mathcal{F}$ be a subset of $B(X, Y)$. If $\mathcal{F}$ is not uniformly bounded, then there exists a dense subset D of X such that for every $x \in D, \{F(x) : F \in \mathcal{F}\}$ is not bounded in Y. (4)

4. a) Read the proof of the closed graph theorem carefully and explain where and how we have used the following facts in the proof. (6)

   i) X is a Banach space.

ii) Y is a Banach space.

iii) F is a closed map.

iv) Which property of continuity is being established to conclude that F is continuous.

b) Which of the following maps are open? Give reasons for your answer. (4)

i) $T:\mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z)=(x,z)$.

ii) $T:\mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x,y,z)=(x,y,0)$.

5. a) Let $f:C[0,1]\to\mathbb{R}$ be given by $f(x)=x(1)\forall x\in C[0,1]$. Show that f is continuous w.r.t the supnorm and f is not continuous w.r.t the p-norm. (5)

b) Let X be an inner product space and $x, y \in X$. Prove that $x \perp y$ if and only if $\|kx+y\|^2 = \|kx\|^2 + \|y^2\|, k \in K$. (5)

6. a) Let $H = R^3$ and F be the set of all $\mathbf{x}=(x_1,x_2,x_3)$ in H such that $x_1 = 0$. Find $F^{\perp}$. Verify that every $\mathbf{x}\in H$ can be expressed as $\mathbf{x}=\mathbf{y}+\mathbf{z}$ where $\mathbf{y}\in F$ and $\mathbf{z}\in F^{\perp}$. (5)

b) Given an example of an Hilbert space H and an operator A on H such that $\sigma_e(A)$ is empty. Justify your choice of example. (2)

c) Let A be a normal operator on a Hilbert space X. Show that $\sigma(A)\subset\sigma_a(A)$ where $\sigma_a(A)$ denotes the approximate eigen spectrum of A and $\sigma(A)$ denotes the spectrum of A. (3)

7. a) Let $X = c_{00}$ with $\|.\|_p$. Give an example of a Cauchy sequence in X that do not converge in X. Justify your choice of example. (4)

b) Give one example of each of the following. Also justify your choice of example. (4)

i) A self-adjoint operator on ℓ^2.

ii) A normal operator on a Hilbert space which is not unitary.

c) Let X be a normed space and Y be proper subspace of X. Show that the interior Y^0 of Y is empty. (2)

8. a) Let X,Y be normed spaces and suppose BL(X,Y) and CL(X,Y) denote, respectively, the space of bounded linear operators from X to Y and the space of compact linear operators from X to Y. Show that CL(X,Y) is linear subspace of BL(X,Y). Also, Show that if Y is a Banach space, then CL(X,Y) is a closed subspace of BL(X,Y). (5)

b) Define a Hilbert-Schmidt operator on a Hilbert space H and give an example. Is every Hilbert-sehmidt operator a compact operator? Justify your answer. (5)

9. a) Let $\{A_n\}$ be a sequence of unitary operators in BL(H). Prove that if $\|A_n - A\| \to 0, A \in BL(H)$, then A is unitary. (4)

b) Define the spectral radius of a bounded linear operator $A \in BL(X)$. Find the spectral radius of A in $BL(\mathbb{R}^3)$, where A is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$. (4)

c) Let X be a Banach space and Y be a closed subspace of X. Let $\pi : X \to X/Y$ be canonical quotient map. Show that π is open. (2)

10. a) Give an example of a compact linear map on l^2. (3)

b) Give an example of a positive operator on $(C^n, \|.\|_2)$. (2)

c) Prove the following result: (5)

Suppose A is a non-zero compact self-adjoint operator on a Hilbert space H over K. Prove that there exists a finite set $\{r_1, r_2, ..., r_n\}$ of a non-zero real numbers with $|r_1| \geq |r_2| \geq |r_3| \geq ... \geq |r_n|$ and an orthonormal set $\{w_1, w_2, ..., w_n\}$ in H such that

$$A(x) = \sum_{i=1}^{n} r_i < x, w_i > w_i, x \in H.$$

Further, mention in which step of the proof it is used that A is a compact self-adjoint operator. Explain why?