MMT-004

ASSIGNMENT BOOKLET

(Valid from 1st January, 2024 to 31st December, 2024)

M.Sc. (Mathematics with Applications in Computer Science)

REAL ANAYLSIS (MMT-004)

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2024)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective courses that we sent you after your enrolment. A weightage of 20 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO. ………………………….…

STUDY CENTRE: ………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved..
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2024. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by December 2024, then you need to get the assignment for the year 2025 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment (MMT – 004)

Course Code: MMT-004
Assignment Code: MMT-004/TMA/2024
Maximum Marks: 100

1. a) Let $X = C[0, 1]$. Define $d : X \times X \to \mathbf{R}$ by $d(f, g) = \int_0^1 | f(t) - g(t) | \, dt, \; f, g \in X$ where the integral is the Riemann integral. Show that d is a metric on X. Find $d(f, g)$ where $f(x) = 4x$ and $g(x) = x^3, \; x \in [0,1]$. (5)

   b) Let (X, d) be a metric space and $a \in X$ be a fixed point of X. Show that the function $f_a : X \to \mathbf{R}$ given by $f_a(x) = d(x, a)$ is continuous. Is it uniformly continuous? Justify you answer. (5)

2. a) Let A and B be any two subsets of a metric space (X, d), then show that
   i) $\text{int}\, A = \cup\{E : \text{ is open and } E \subseteq A\}$
   ii) $\text{int}(A \cap B) = \text{int}\, A \cap \text{int}\, B$
   iii) $\text{int}(A \cup B) \supseteq \text{int}\, A \cap \text{int}\, B$
   iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. (6)

   b) Find the interior, boundary and closure of the following sets A in $\mathbf{R}$ with the usual metric and discrete metric.
   i) $A = \mathbf{Q}$, the set of rationals in $\mathbf{R}$
   ii) $A =]1, 2] \cup]2, 4[$ (4)

3. a) Let (X, d_1) and (Y, d_2) be metric spaces. Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ where A is any subset of X (4)

   b) Let (X_1, d_1) and (X_2, d_2) be two discrete metric spaces. Verify that the product metric on $X_1 \times X_2$ is discrete. (3)

   c) Show that an infinite discrete metric space X is bounded but not totally bounded. (3)

4. a) Find the first derivative $f'(\mathbf{a})$ of the function f defined by $f : \mathbf{R}^3 \to \mathbf{R}^2$ given by $f(x, y, z) = (xyz, \; x + y + z^2)$ where $\mathbf{a} = (1. -1, 2)$. (3)

   b) Let E be an open subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$ be a function such that each of its components function f_i are differentiable, then show that f is differentiable. Is the converse of this result true? Justify your answer. (4)

   c) Near what points may the surface $z^2 + xz + y = 0$ be represented uniquely as a graph of a differentiable function $z = k(x, y)$? Locate such a point. (3)

5. a) Use the method of Lagrange's multiplier method to find the shortest possible distance from the ellipse $x^2 + 2y^2 = 2$ to the line $x + y = 2$. (6)

b) Find the directional derivative of the function $f : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$f(x, y, z, w) = (x^2y, xyz, x^2 + y^2 + zw^2)$$

at $a = (1, 2, -1, -2)$ in the direction $v = (0, 1, 2, -2)$. (4)

6. a) Let A be a compact non-empty subset of a metric space (X, d) and let F be a closed subset of X such that $A \cap F = \phi$, then show that $d(A, F) > 0$ where $d(A,F) = \inf\{d(a, b) : a \in A, b \in F\}$. (3)

b) Give an example of the following with justification (4)

i) A vector-valued function $f : \mathbf{R}^3 \to \mathbf{R}^3$ which is not differentiable at $(0,0,0)$.

ii) A function which is Legesgue measurable on $\mathbf{R}$.

c) Show that the components of a metric space is either identical or pairwise disjoint. (3)

7. a) Let $\mathbf{Q}$ be the set of rationals with the metric defined on $\mathbf{Q}$ by $d : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}$, defined by $d(x, y) = |x - y|, \forall x, y \in \mathbf{R}$.

Show that $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is Cauchy sequence in **Q,** but does not converge in $\mathbf{Q}$ and $\left\{\frac{1}{3^n}\right\}$ is a Cauchy sequence $\mathbf{Q}$ which converges in $\mathbf{Q}$ to the limit 0. (5)

b) Which of the following sets are totally bounded? Give reasons for your answer. Are they compact? (3)

i) $2\mathbf{N}$ in $(\mathbf{N}, d)$ where d is the discrete metric.

ii) $[0,2] \cup [5,10]$ in $(\mathbf{R}, d)$ where d is the Euclidean metric.

c) Which of the following sets are connected sets in $\mathbf{R}^2$ with the metric given against it? Justify your answer.

i) $A = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$ under the standard metric.

ii) $A = \{(x, y) : x^2 + y^2 = 1\}$ under the discrete metric. (2)

8. a) Consider $\mathbf{Z}$ and let $\mathcal{F}_1$ denote the class of subsets of $\mathbf{Z}$, given by $\mathcal{F}_1 = \{A \subset \mathbf{Z}$: either A is finite or A^c is finite$\}$. Check whether $\mathcal{F}_1$ is a σ algebra or not. (3)

b) Let A be any set in $\mathbf{R}$, show that $m^*(A) = m^*(A + x)$ where m^* denotes the outer measure. (4)

c) Find the measure of the following sets.

i) $E = \bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, b\right)$

ii) $E = \mathbf{Q} \cup \{1, 2, 3, 4\}$

iii) $E =]5, 7[\cup [7, 7.5]$. (3)

9. a) Show that if f is measurable, then the function $f^a(x)$ given by

$$f^a(x) = \begin{cases} a & \text{if } f(x) > a \\ f(x) & \text{if } f(x) \le a \end{cases}$$

is also measurable. (3)

b) Verify Bounded Convergence Theorem for the sequence of functions $\{f_n\}$ where

$$f_n(x) = \frac{1}{(1 + x/n)^n}, \; 0 \le x \le 1, \; n \in \mathbf{N}$$ (4)

c) Find the fourier series of the function f defined by (3)

$$f(x) = \begin{cases} -x^2, & -\pi < x \le 0 \\ x^2, & 0 < x < \pi \end{cases}$$

10. State whether the following statements are True or False. Justify your answers. (10)

a) The sequence $\left\{\left(\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbf{N}\right\}$ is convergent in $\mathbf{R}^2$ under the discrete metric on $\mathbf{R}^2$.

b) A subset in a metric space is compact if it is closed.

c) Continuous image of a path connected space is path connected.

d) The second derivative of a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ never vanishes.

e) If $\int_A f \, dm = \int_A g \, dm$ for all $A \in \mathcal{M}$, then $f = g$.